PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 1, 1986

ESTIMATES FOR PARTIAL SUMS OF CONTINUED FRACTION PARTIAL QUOTIENTS

HAROLD G. DIAMOND AND JEFFREY D. VAALER

Metric-type estimates are given for a class of partial sums involving continued fraction partial quotients. These results extend a well known theorem of Khinchin and yield an almost-everywhere estimate for the quantity in the title.

1. Introduction. For α an irrational number in $(0,1)$ let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = \langle 0, a_1, a_2, \dots \rangle$$

be the representation of α as a regular continued fraction ([**4**, Ch. X], [**5**]). The numbers $a_n = a_n(\alpha)$ are called the *partial quotients* of α.

A well-known theorem of Khinchin [**5**], [**6**] in the metric theory of continued fractions asserts that if F is an arithmetic function satisfying $F(r) \ll r^{1/2-\delta}$ for some $\delta > 0$ and if $S_N(F,\alpha) := F(a_1(\alpha)) + \cdots + F(a_N(\alpha))$ for each positive integer N, then

$$\lim_{N\to\infty} \frac{1}{N} S_N(F,\alpha) = \frac{1}{\log 2} \sum_{r=1}^{\infty} F(r) \log\left\{1 + \frac{1}{r(r+2)}\right\} \tag{1}$$

holds for almost all α in $(0,1)$. This result has been extended by others ([**2**, §4], [**7**, Theorem 4]). In particular, we note that the Birkhoff Ergodic Theorem implies that (1) holds if its right-hand side is absolutely convergent.

Here we shall establish analogues of (1) for arithmetic functions F which grow more rapidly than is allowed by Khinchin's theorem. In particular we shall consider the case $F(r) = I(r) = r$ and estimate

$$S_N(I,\alpha) = a_1(\alpha) + \cdots + a_N(\alpha).$$

Khinchin noted at the end of his book *Continued Fractions* [**5**] that $S_N(I,\alpha)/N$ could not have a finite limit for most values of α. Indeed, for almost all α the inequality $a_n(\alpha) > n \log n$ holds for an infinite sequence of integers n in consequence of the following result of Borel and Bernstein