PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 2, 1986

STOCHASTIC INTEGRATION IN FOCK SPACE

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In this paper, using purely Hilbert space-theoretic methods, an analogue of the Itô integral is constructed in the symmetric Fock space of a direct integral $\mathfrak{H}$ of Hilbert spaces over the real line. The classical Itô integral is the special case when $\mathfrak{H} = L^2[0, \infty)$. An explicit formula is obtained for the projection onto the space of 'non-anticipating functionals', which is then used to prove that simple non-anticipating functionals are dense in the space of all non-anticipating functionals. After defining the analogue of the Itô integral, its isometric nature is established. Finally, the range of this 'integral' is identified; this last result is essentially the Kunita-Watanabe theorem on square-integrable martingales.

Preliminaries. (a) *Symmetric Fock space*: If $\mathfrak{H}$ is a (complex) Hilbert space, the symbol $\mathfrak{H}^{(s)n}$ will denote the Hilbert space of symmetric tensors of rank n; alternatively, $\mathfrak{H}^{(s)n}$ is the closed subspace of $\otimes^n \mathfrak{H}$ spanned by $\{x \otimes \cdots \otimes x \colon x \in \mathfrak{H}\}$. (In the sequel, the symbol $\operatorname{sp} S$ will denote the closed subspace spanned by the set S of vectors.) By convention, $\mathfrak{H}^{(s)0} = \mathbf{C}$. We shall also write $\otimes^n x$ for $x \otimes \cdots \otimes x$, with the convention that $\otimes^0 x = 1$.

The symmetric Fock space over $\mathfrak{H}$, is by definition, the Hilbert direct sum

$$\Gamma(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{(s)n}.$$

If $x \in \mathfrak{H}$, then $\Gamma(x)$ will denote the 'exponential' vector in $\Gamma(\mathfrak{H})$ defined by

$$\Gamma(x) = \left(1, x, \frac{\otimes^2 x}{\sqrt{2!}}, \ldots, \frac{\otimes^n x}{\sqrt{n!}}, \ldots\right).$$

The following are easily verified:

$$\text{(1)}\quad \begin{aligned} &\text{(i)} && \Gamma(\mathfrak{H}) = \operatorname{sp}\{\Gamma(x) \colon x \in \mathfrak{H}\}; \\ &\text{and} && \\ &\text{(ii)} && \langle \Gamma(x), \Gamma(y) \rangle = \exp\langle x, y \rangle, \qquad x, y \in \mathfrak{H}. \end{aligned}$$