PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 2, 1986

ASYMPTOTIC EXPANSIONS OF THE LEBESGUE CONSTANTS FOR JACOBI SERIES

C. L. FRENZEN AND R. WONG

Explicit expressions are obtained for the implied constants in the two O-terms in Lorch's asymptotic expansions of the Lebesgue constants associated with Jacobi series [Amer. J. Math., 81 (1959), 875–888]. In particular, a question of Szegö concerning asymptotic monotonicity of the Lebesgue constants for Laplace series is answered. Our method differs from that of Lorch, and makes use of some recently obtained uniform asymptotic expansions for the Jacobi polynomials and their zeros.

1. Introduction and summary. The nth partial sum of the Fourier series of an arbitrary function can be written in the form of an integral involving the Dirichlet kernel. The integral of the absolute value of this kernel is known as the nth *Lebesgue constant*, and is usually denoted by

$$L_n = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{\sin(t/2)}\,dt; \tag{1.1}$$

see [**19**, p. 172]. The behavior of the sequence $\{L_n\}$ is closely connected with convergence and divergence properties of Fourier series, and the importance of this sequence has led many mathematicians to be concerned not only with just its asymptotic formula but also with its full asymptotic expansion. First, Fejer [**1**] showed that

$$L_n = \frac{4}{\pi^2}\log n + c_0 + \frac{c_1}{n} + \frac{\alpha(n)}{n^2}, \tag{1.2}$$

where c_0 and c_1 are constants and $\alpha(n) = O(1)$ as $n \to \infty$. An explicit expression was given for c_0 but not for c_1. Later, infinite asymptotic expansions were derived by Gronwall [**4**], Watson [**18**] and Hardy [**7**].

In an entirely analogous manner, the nth partial sum of the expansion of an arbitrary function in terms of Jacobi polynomials can be written as an integral involving a kernel; see, e.g., [**17**, p. 39]. The nth Lebesgue constant in this case has the integral representation

$$L_n(\alpha,\beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \cdot \int_0^{\pi} \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} \left|P_n^{(\alpha+1,\beta)}(\cos\theta)\right| d\theta. \tag{1.3}$$