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ON EXTREME POINTS AND SUPPORT POINTS OF THE FAMILY OF STARLIKE FUNCTIONS OF ORDER α

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Let $\mathrm{St}(\alpha)$ denote the subclass of functions $f(z)$ analytic in the open unit disk D which satisfy the conditions $f(0) = 0$, $f'(0) = 1$ and $\mathrm{Re}(zf'(z)/f(z)) > \alpha$ for z in D. In this note we investigate the compact, convex family $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ which is the closed convex hull of the set of all functions analytic in D that are subordinate to some function in $\mathrm{St}(\alpha)$, $\alpha < 1/2$. The principal result establishes that every support point of $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ arising from a "nontrivial" functional must also be an extreme point, hence a function of the form $f(z) = xz/(1 - yz)^{2(1-\alpha)}$, $|x| = |y| = 1$.

To amplify on this synopsis, let $\mathscr{A}$ denote the set of functions analytic in the open unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$. Then $\mathscr{A}$ is a locally convex linear topological space under the topology of uniform convergence on compact subsets of D. A function f in $\mathscr{A}$ is said to be subordinate to a function F in $\mathscr{A}$ (written $f \prec F$), if there is a function φ in B_0 such that $f(z) = F(\varphi(z))$, where $B_0 = \{\varphi \in \mathscr{A} \mid \varphi(0) = 0, |\varphi(z)| < 1 \text{ in } D\}$.

Let $\mathscr{F}$ be a compact subset of $\mathscr{A}$. A function f in $\mathscr{F}$ is a support point of $\mathscr{F}$ if there is a continuous linear functional J on $\mathscr{A}$ such that

$$\mathrm{Re}\, J(f) = \max\{\mathrm{Re}\, J(g) \mid g \in \mathscr{F}\}$$

and $\mathrm{Re}\, J$ is non-constant on $\mathscr{F}$. We use $\Sigma\mathscr{F}$ to denote the set of support points of $\mathscr{F}$ and $\overline{\mathrm{co}}\mathscr{F}$ and $\mathscr{E}\overline{\mathrm{co}}\mathscr{F}$ to denote, respectively, the closed convex hull of $\mathscr{F}$ and the set of extreme points of the closed convex hull of $\mathscr{F}$.

Let $S(\mathrm{St}(\alpha))$ denote the set of functions in $\mathscr{A}$ that are subordinate to some function in $\mathrm{St}(\alpha)$. Then $S(\mathrm{St}(\alpha))$ is a compact subset of $\mathscr{A}$ [**11**, p. 365]. In [**3**] and [**6**] it was shown that

$$\overline{\mathrm{co}}\, \mathrm{St}(\alpha) = \left\{ \int \frac{z}{(1 - xz)^{2(1-\alpha)}}\, d\mu(x) : \mu \text{ is a probability measure the unit circle} \right\}$$

and that

$$\mathscr{E}\overline{\mathrm{co}}\, \mathrm{St}(\alpha) = \sum \mathrm{St}(\alpha) = \left\{ \frac{z}{(1 - xz)^{2(1-\alpha)}} : |x| = 1 \right\}.$$