PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 1, 1986

ON CONTINUOUS APPROXIMATIONS FOR MULTIFUNCTIONS

F. S. DE BLASI AND J. MYJAK

Problems concerning the approximation of convex valued multifunctions by continuous ones are considered. Approximation results of the type obtained by Gel′man, Cellina, and Hukuhara for Pompeiu-Hausdorff upper semicontinuous multifunctions are shown to hold for some larger classes of multifunctions. Moreover, it is proved that Pompeiu-Hausdorff semicontinuous multifunctions, with convex bounded values, are continuous almost everywhere (in the sense of the Baire category). As an application, an alternative proof is given of Kenderov's theorem stating that a maximal monotone operator is almost everywhere single-valued.

1. Introduction and preliminaries. Let X be a metric space. Let Y be a normed space. Denote by $\mathscr{C}(Y)$ (resp. $\mathscr{C}_b(Y)$, $\mathscr{C}_k(Y)$) the class of all nonempty subsets of Y which are convex (resp. convex bounded, convex compact). In any metric space, $S(u,r)$ stands for the open ball around u with radius $r > 0$.

We shall consider the following approximation problems (for the terminology see below):

I. Given a multifunction F: $X \to \mathscr{C}_b(Y)$ and an $\varepsilon > 0$, find an h-continuous multifunction G: $X \to \mathscr{C}_b(Y)$ such that h(graph G, graph F) $\leq \varepsilon$ (where h denotes the Pompeiu-Hausdorff pseudometric).

II. Given a multifunction F: $X \to \mathscr{C}(Y)$ and an $\varepsilon > 0$, find a continuous single-valued function g: $X \to Y$ such that h^*(graph g, graph F) $\leq \varepsilon$ (where h^* denotes the separation function).

III. Given a multifunction F: $X \to \mathscr{C}_b(Y)$, find a sequence $\{G_n\}$ of h-continuous multifunctions G_n: $X \to \mathscr{C}_b(Y)$ satisfying for each $x \in X$, $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$, and $G_n(x) \supset F(S(x, \sigma_n(x)))$ for some $\sigma_n(x) > 0$.

Apparently, the idea of constructing continuous approximations for a multifunction goes back to Von Neumann [**29**]. When F is upper semicontinuous in the sense of the Pompeiu-Hausdorff separation h^* "h^*-u.s.c.", the approximation problems I, II, and III have been investigated by Gel′man (see references in [**2**]), Cellina [**5**, **6**], and Hukuhara [**17**], respectively. Further results can be found in [**23**], [**24**], [**13**], [**8**].