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ASYMPTOTIC BEHAVIOR OF TWO SEMI-LINEAR ELLIPTIC FREE BOUNDARY PROBLEMS

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Given a bounded open set $\Omega \subseteq \mathbf{R}^n$ with $C^{2+\alpha}$ boundary and a monotone increasing function $f(t)$ with $f(0) = 0$, this paper treats two related exterior free boundary problems:

Problem A: Given $\lambda > 0$, determine $u \in C_0^{2+\alpha_1}(\mathbf{R}^n - \Omega)$ satisfying:

$$\Delta u = \lambda f(u) \quad \text{in } \mathbf{R}^n - \bar{\Omega}, \qquad u = 1 \quad \text{on } \partial\Omega. \tag{1.1}$$

Problem B: Given $c > 0$, determine $v \in C_0^{2+\alpha_1}(\mathbf{R}^n - \Omega)$ satisfying:

$$\Delta v = f(v) \quad \text{in } \mathbf{R}^n - \bar{\Omega}, \qquad v = c \quad \text{on } \partial\Omega. \tag{1.2}$$

In both problems, the free boundary is the boundary of the support of the sought function.

Problem A comes from the Langmuir-Hinshelwood model for chemical kinetics (among biochemists this is known as Michaelis-Menten kinetics) ([**1**], [**2**]). Ω then represents a patch of constant concentration of a reactant diffusing into a substrate. Problem B has appeared in a paper by Caffarelli and Spruck [**3**], in which they show that if Ω is convex, then the level surfaces of u are convex surfaces. A special case of Problem B appears in continuous hot-dip galvanizing ([**9**]).

Section 2 is concerned about existence and uniqueness for fixed c or λ. From this point of view, Problem A is contained in Problem B, therefore in this section we deal only with Problem B. The results follow easily from the work that has been done on the interior problem ([**5**], [**7**], [**8**]). With this as a starting point, we attempt to determine characteristics of the free boundary. The main thrust of this paper is to show that in R^2, as $\lambda \to 0$ (Problem A) or as $c \to \infty$ (Problem B), the free boundaries are asymptotic to a family of circles. By this we mean the following. Let p be a point in Ω, let $d(p)$ be the distance from p to the point on the free boundary closest to p, and let $d_1(p)$ be the distance from p to the point on the free boundary farthest from p. Then as $\lambda \to 0$ (Problem A) or as $c \to \infty$ (Problem B), the ratio $d_1(p)/d(p)$ approaches 1. Thus, if we scale the picture so that the point on the free boundary closest to p is at distance 1 from p, then the free boundary in this scaled