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GENERALIZED s-NUMBERS OF τ-MEASURABLE OPERATORS

THIERRY FACK AND HIDEKI KOSAKI

We give a self-contained exposition on generalized s-numbers of τ-measurable operators affiliated with a semi-finite von Neumann algebra. As applications, dominated convergence theorems for a gage and convexity (or concavity) inequalities are investigated. In particular, relation between the classical L^p-norm inequalities and inequalities involving generalized s-numbers due to A. Grothendieck, J. von Neumann, H. Weyl and the first named author is clarified. Also, the Haagerup L^p-spaces (associated with a general von Neumann algebra) are considered.

0. Introduction. This article is devoted to a study of generalized s-numbers of τ-measurable operators affiliated with a semi-finite von Neumann algebra. Also dominated convergence theorems for a gage and convexity (or concavity) inequalities are investigated.

In the "hard" analysis of compact operators in Hilbert spaces, the notion of s-numbers (singular numbers) plays an important role as shown in [**10**], [**24**]. For a compact operator A, its nth s-number $\mu_n(A)$ is defined as the nth largest eigenvalue (with multiplicity counted) of $|A| = (A^*A)^{1/2}$. The following expression is classical:

$$\mu_n(A) = \inf\{\|AP_{\mathscr{K}}\|;\ \mathscr{K} \text{ is a closed subspace with } \dim \mathscr{K}^{\perp} \le n\},$$

where $P_{\mathscr{K}}$ denotes the projection onto $\mathscr{K}$.

In the present article, we will study the corresponding notion for a semi-finite von Neumann algebra. More precisely, let $\mathscr{M}$ be a semi-finite von Neumann algebra with a faithful trace τ. For an operator A in $\mathscr{M}$, the "tth" generalized s-number $\mu_t(A)$ is defined by

$$\mu_t(A) = \inf\{\|AE\|;\ E \text{ is a projection in } \mathscr{M} \text{ with } \tau(1-E) \le t\}, \quad t > 0.$$

Notice that the parameter t is no longer discrete corresponding to the fact that τ takes continuous values on the projection lattice. Actually this notion has already appeared in the literature in many contexts ([**8**], [**11**], [**25**], [33]). In fact, Murray and von Neumann used it (in the II_1-case), [**18**]. We will consider generalized s-numbers of τ-measurable operators in the sense of Nelson [**19**]. This is indeed a correct set-up to consider generalized s-numbers. In fact, the τ-measurability of an operator A