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A LOOP SPACE WHOSE HOMOLOGY HAS TORSION OF ALL ORDERS

DAVID J. ANICK

In this note we will construct a simply-connected finite CW complex X of dimension four with the following property. For each positive integer m, there is an element ω_m of order m in $\pi_{m+1}(\Omega X)$ whose Hurewicz image $h(\omega_m) \in H_{m+1}(\Omega X)$ also has order m. Such a space illustrates how intricate loop space homology can be even for relatively simple spaces.

Our approach is first to construct a certain non-commutative graded Hopf algebra over **Z**, which we call A. Next we will realize A topologically by a simply-connected four-dimensional space X in the sense that A will be the image in $H_*(\Omega X)$ of $H_*(\Omega X^2)$, where X^2 is the 2-skeleton of X. The desired properties of X will then follow from the properties of A. We will use the very simple commuting diagram

$$\begin{array}{ccccccc} \pi_n(X^2) & \xrightarrow[\theta_2]{\approx} & \pi_{n-1}(\Omega X^2) & \xrightarrow[h_2]{} & H_{n-1}(\Omega X^2) & \searrow & \\ \eta_\# \downarrow & & (\Omega\eta)_\# \downarrow & & (\Omega\eta)_* \downarrow & & A \\ \pi_n(X) & \xrightarrow[\theta]{\approx} & \pi_{n-1}(\Omega X) & \xrightarrow[h]{} & H_{n-1}(\Omega X). & \swarrow & \end{array} \tag{1}$$

Here $\eta\colon X^2 \to X$ is the inclusion; h and h_2 denote Hurewicz homomorphisms; and θ and θ_2 are the familiar isomorphisms.

For any ring R, let $R\langle y_1, \ldots, y_n\rangle$ denote the free associative algebra over R with (non-commuting) generators $y_1, \ldots, y_n$. The R-algebra $R\langle y_1, \ldots, y_n\rangle$ becomes a (non-negatively) graded ring by allowing each y_i to have degree one and assigning all of R to degree zero. If T is a graded ring ("graded" here will always mean graded by the non-negative integers), write $|x|$ for the degree of a homogeneous non-zero element x. The *commutator* of x and y in T is

$$[x, y] = xy - (-1)^{|x|\cdot|y|}(yx).$$

The Hopf algebra A we seek will be constructed as the semi-tensor product [**8**] or smash product [**4**] of the free Hopf algebra $\mathbf{Z}\langle x_1, x_2\rangle$ with