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THE SPACING OF THE MINIMA IN CERTAIN CUBIC LATTICES

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Let $\mathcal{K}$ be a cubic field with negative discriminant; let $\mu, \nu \in \mathcal{K}$; and let $\mathcal{R}$ be a lattice with basis $\{1, \mu, \nu\}$ such that 1 is a minimum of $\mathcal{R}$. If

$$1 = \theta_1, \theta_2, \theta_3, \ldots, \theta_n, \ldots$$

is a chain of adjacent minima of $\mathcal{R}$ with $\theta_{i+1} > \theta_i$ ($i = 1, 2, 3, \ldots$), then

$$\theta_{n+5} \geq \theta_{n+3} + \theta_n.$$

This result can be used to prove that if p is the period of Voronoi's continued fraction algorithm for finding the fundamental unit ε_0 of $\mathcal{K}$, then

$$\varepsilon_0 > \tau^{p/2},$$

where $\tau = (1 + \sqrt{5})/2$. It is also shown that

$$\theta_n > 4^{[(n-1)/7]}.$$

1. Introduction. In order to discuss the problems considered in this paper, it is necessary to give a brief description of the properties of cubic lattices. For a more extensive and more general treatment of these topics we refer the reader to Delone and Faddeev [**1**].

Let $f(x) \in \mathbf{Z}[x]$ be a cubic polynomial, irreducible over the rationals $\mathcal{Q}$ and having a negative discriminant. Let δ be the real zero of $f(x)$ and denote by $\mathcal{K} = \mathcal{Q}(\delta)$ the complex cubic field formed by adjoining δ to $\mathcal{Q}$. If $\mathcal{E}_3$ denotes Euclidean 3-space, we can associate with each $\alpha \in \mathcal{K}$ a point $A \in \mathcal{E}_3$, where

$$A = \left(\alpha, (\alpha' - \alpha'')/2i, (\alpha' + \alpha'')/2\right),$$

$i^2 + 1 = 0$, and α', α'' are the conjugates of α. Since $f(x)$ has a negative discriminant, all three components of A must be real. If $\lambda, \mu, \nu \in \mathcal{K}$ and λ, μ, ν are rationally independent, we define the cubic lattice $\mathcal{L}$ by

$$\mathcal{L} = \left\{ u\Lambda + vM + wN \mid (u, v, w) \in \mathbf{Z}^3 \right\}.$$

We say that $\mathcal{L}$ has a basis $\{\lambda, \mu, \nu\}$ and denote $\mathcal{L}$ by $\langle \lambda, \mu, \nu \rangle$. For the sake of convenience we will often use the expression $\alpha \in \mathcal{L}$ to denote that it is the corresponding point $A \in \mathcal{E}_3$ that is actually in $\mathcal{L}$. Also, if $\mathcal{L} = \langle \lambda, \mu, \nu \rangle$, we define $\alpha\mathcal{L}$ ($\alpha \in \mathcal{K}$) to be the lattice $\langle \alpha\lambda, \alpha\mu, \alpha\nu \rangle$.