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INCREASING PATHS ON THE ONE-SKELETON OF A CONVEX COMPACT SET IN A NORMED SPACE

LEONI DALLA

Let C be a convex compact set in a normed space E and let $\text{skel}_1 C$ be the subset of C that contains those boundary points of C which are not centres of 2-dimensional balls in C. When l is a continuous functional on E, we say that the path $P = g([\alpha, \beta])$ is l-strictly increasing if $l(g(t_1)) < l(g(t_2))$ for every t_1, t_2 such that $\alpha \leq t_1 < t_2 \leq \beta$. D. G. Larman proved the existence of an l-strictly increasing path on the one skeleton of C with $l(g(\alpha)) = \min_{x \in C} l(x)$ and $l(g(\beta)) = \max_{x \in C} l(x)$.

In this paper we prove a theorem concerning the number of l-strictly increasing paths on the one-skeleton of C, that are mutually disjoint and along each of which l assumes values in a range arbitrarily close to its range on C.

1. The results.

We quote and prove the following theorem

THEOREM 1. *Let C be a compact convex set of infinite dimension in a normed space E and l be a continuous linear functional on E, which is non constant on C. Let $\varepsilon > 0$ be given, $M = \max_{x \in C} l(x)$ and $m = \min_{x \in C} l(x)$. Then, for every $n = 1, 2, 3, \ldots$ there exist n l-strictly increasing paths, $P_k = g_k([\alpha, \beta])$, $k = 1, 2, \ldots, n$ on the one-skeleton of C, such that* $\operatorname{rel\,int} P_i \cap \operatorname{rel\,int} P_j = \varnothing$ *with $i \neq j$, $l(g_k(\alpha)) = m + \varepsilon$ and $l(g_k(\beta)) = M - \varepsilon$ for $k = 1, 2, \ldots, n$.*

Proof. Consider the sets $K_0 = \{x \in C\colon l(x) = M - \varepsilon\}$ and $K_1 = \{x \in C\colon l(x) = m - \varepsilon\}$. These sets are of infinite dimension and lie on two parallel hyperplanes. We define

$$A = C \cap \{x \in E\colon l(x) \geq m + \varepsilon\} \cap \{x \in E\colon l(x) \leq M - \varepsilon\}$$

Then we may select n linearly independent vectors $e_1, e_2, \ldots, e_n$ and n linear functionals $l_1 = l, l_2, \ldots, l_n$ on E such that the following properties hold:

(i) $l_1(e_1) = 1$, $l_i(e_i) \neq 0$ for $i = 2, 3, \ldots, n$ and $l_i(e_j) = 0$ for $i \neq j$

(ii) Let $E_n = [e_1, e_2, \ldots, e_n]$ be the n-dimensional subspace of E spanned by $e_1, e_2, \ldots, e_n$ and π_0 be the projection map on E, defined by $\pi_0(x) = l_1(x)e_1 + \cdots + l_n(x)e_n$. Then $\dim \pi_0(K_0) = \dim \pi_0(K_1) = n - 1$.