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ORDER IDEALS IN CATEGORIES

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In the program to develop enriched category theory in a topos $\mathscr{E}$ it seems worthwhile to study the two particular bases Ω and $\mathbf{R}^+$; that is, the ordered objects of truth values and of non-negative extended reals with their appropriate monoidal structures. Categories in $\mathscr{E}$ enriched in Ω are ordered objects in $\mathscr{E}$, and it is this example we wish to study here.

Categories in $\mathscr{E}$ enriched in $\mathbf{R}^+$ are metric spaces in $\mathscr{E}$ [**8**] and the relevant $\mathbf{R}^+$ has been studied in [**10**]. Since ordered objects occur at the very foundations of elementary topos theory, they have already been extensively studied (especially by Mikkelsen [**9**] and Brook [**3**]). However, our purpose is to emphasize the enriched-category viewpoint to give a guide to further development of the program.

Ordered objects can be defined without Ω, of course, and much of the theory can be developed in a category $\mathscr{E}$ much more general than a topos. Our first two sections take this general approach. The first section deals with *ideals* in a regular category; from the enriched-category viewpoint these are the *modules* (= bimodules = profunctors = distributors). There is a bicategory $\mathrm{Idl}(\mathscr{E})$ of ordered objects and ideals. The first key result is that an ideal has a right adjoint if and only if it is locally principal. This means that locally principal ideals play the role that cauchy sequences play in metric space theory [**8**]. The question of whether every ordered object is "cauchy complete" thus becomes the question of whether locally principal implies principal. We show that this is true precisely when $\mathscr{E}$ satisfies the axiom of choice. The remainder of the first section deals with completeness of ordered objects.

The purpose of the second section is to construct, for ordered objects A, B, an object $[A, B]^*$ of order-preserving arrows from A to B with right adjoints and an object $[A, B]^{**}$ of order-preserving arrows from A to B with right adjoints which have right adjoints. This requires $\mathscr{E}$ to be cartesian closed.

For the final section, $\mathscr{E}$ is required to be an elementary topos. For an ordered object A, we construct the object $\mathscr{P}A$ of order ideals in A which, in enriched-category terms, is the object appropriate for receiving the yoneda embedding. After developing sufficiently the properties of $\mathscr{P}A$, we