PACIFIC JOURNAL OF MATHEMATICS
Vol. 125, No. 1, 1986

USING PREDICTION PRINCIPLES TO CONSTRUCT ORDERED CONTINUA

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In this paper, we show that the elements of a $\diamondsuit$-sequence can be ordered lexicographically to produce an ordered continuum. An application of this idea answers a question of V. Malyhin and others: Is there a compact Hausdorff space in which no two points have equal character? We show that the consistency strength of the existence of such a space lies between that of an inaccessible and a Mahlo cardinal. We show that compactness is essential in this result by constructing, in ZFC, a σ-compact Hausdorff space in which no two points have equal character.

Let us begin with some definitions:

DEFINITION 1. $\{f_\alpha\colon \alpha \in E\}$ is a *$\diamondsuit_\kappa(E)$-sequence* (where $E \subset \kappa - \{0\}$ and $f_\alpha\colon \alpha \to 2$) if, for each $f\colon \kappa \to 2$, there is $\alpha \in E$ such that $f_\alpha \subset f$.

This is not exactly the standard definition (we use the characteristic functions of subsets of κ, we trap only once and do not require that E be stationary or even cofinal in κ) but it is equivalent in most cases. The lexicographic ordering is not well defined because their domains are not equal. We need to subtract some f_αs which are "not needed". Let us fix this idea.

DEFINITION 2. $\{f_\alpha\colon \alpha \in E\}$ is a *minimal* $\diamondsuit_\kappa(E)$-sequence if, whenever $F \subset E$ and $\{f_\alpha\colon \alpha \in F\}$ is a $\diamondsuit_\kappa(F)$-sequence, F must equal E.

This seems like a strong condition but it is not. We *can* subtract the f_α's which are not needed.

LEMMA 1. *If $\{f_\alpha\colon \alpha \in E\}$ is a $\diamondsuit_\kappa(E)$-sequence, then there is $F \subset E$ such that*

$$\{f_\alpha\colon \alpha \in F\} \text{ is a minimal } \diamondsuit_\kappa(F)\text{-sequence.}$$

Proof. The idea is to inductively subtract any f_α compatible with f_β when $\beta < \alpha$. That is, $\alpha \in F$ iff, for each $\beta \in \alpha \cap F$, $f_\beta \cup f_\alpha$ is not a function. $\{f_\alpha\colon \alpha \in F\}$ is a $\diamondsuit_\kappa(F)$-sequence since, for each $f\colon \kappa \to 2$ there is a minimal $\alpha \in E$ such that $f \restriction \alpha = f_\alpha$. By the construction of F and the