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WEAK CONVERGENCE AND NON-LINEAR ERGODIC THEOREMS FOR REVERSIBLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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Let S be a semitopological semigroup. Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and $\mathcal{S} = \{T_a;\ a \in S\}$ be a continuous representation of S as nonexpansive mappings of C into C such that the common fixed point set $F(\mathcal{S})$ of $\mathcal{S}$ in C is nonempty. We prove in this paper that if S is right reversible (i.e. S has finite intersection property for closed right ideals), then for each $x \in C$, the closed convex set $W(x) \cap F(\mathcal{S})$ consists of at most one point, where $W(x) = \bigcap\{K_s(x);\ s \in S\}$, $K_s(x)$ is the closed convex hull of $\{T_t x;\ t \geq s\}$ and $t \geq s$ means $t = s$ or $t \in \overline{Ss}$. This result is applied to study the problem of weak convergence of the net $\{T_s x;\ s \in S\}$, with S directed as above, to a common fixed point of $\mathcal{S}$. We also prove that if E is uniformly convex with a uniformly Fréchet differentiable norm, S is reversible and the space of bounded right uniformly continuous functions on S has a right invariant mean, then the intersection $W(x) \cap F(\mathcal{S})$ is nonempty for each $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_s = T_s P = P$ for all $s \in S$ and $P(x)$ is in the closed convex hull of $\{T_s(x);\ s \in S\}$, $x \in C$.

1. Introduction. Let S be a semitopological semigroup i.e. S is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. S is called *right reversible* if any two closed left ideals of S has non-void intersection. In this case, $(S, \leq)$ is a directed system when the binary relation " $\leq$ " on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$, $a, b \in S$. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [**13**, p. 335]). Left reversibility of S is defined similarly. S is called *reversible* if it is both left and right reversible.

Let E be a uniformly convex Banach space and $\mathcal{S} = \{T_s;\ s \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of E into C i.e. $T_{ab}(x) = T_a T_b(x)$, $a, b \in S$, $x \in C$ and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set $\{x \in C;\ T_s(x) = x$ for all $s \in S\}$ of common fixed points of $\mathcal{S}$ in C. Then, as is