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THE ASYMPTOTIC BEHAVIOR OF A FAMILY OF SEQUENCES

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A class of sequences defined by nonlinear recurrences involving the greatest integer function is studied, a typical member of the class being

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \quad \text{for } n \geq 1.$$

For this sequence, it is shown that $\lim a(n)/n$ as $n \to \infty$ exists and equals $12/(\log 432)$. More generally, for any sequence defined by

$$a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor) \quad \text{for } n \geq 1,$$

where the $r_i > 0$ and the m_i are integers ≥ 2, the asymptotic behavior of $a(n)$ is determined.

1. Introduction. Rawsthorne [**R**] recently asked whether the limit $a(n)/n$ exists for the sequence $a(n)$ defined by

$$(1.1) \quad a(0) = 1, \qquad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor), \quad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. If the limit exists, Rawsthorne also asked for its value. We have answered these questions [**EHOPR**]: the limit exists and equals $12/\log 432$, where, as in the rest of the paper, log denotes the natural logarithm. Our method leads to a more general result about such recursively defined sequences.

Let $a(n)$ be the sequence defined by

$$(1.2) \qquad a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1,$$

where $r_i > 0$ and the m_i's are integers ≥ 2. Let τ be the (unique) solution to

$$(1.3) \qquad \sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} = 1.$$

We distinguish two cases: if there is an integer d and integers u_i such that $m_i = d^{u_i}$, we are in the *lattice* case, otherwise we are in the *ordinary* case. In the ordinary case, $\lim a(n)/n^{\tau}$ exists; in the lattice case, $\lim a(n)/n^{\tau}$ does not exist, but $\lim_{k \to \infty} a(d^k)/d^{k\tau}$ exists. The limit in either case is