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EXTENSIONS OF REPRESENTATIONS OF LIE ALGEBRAS

JOHN GERARD RYAN

Let ϕ: $L_1 \to L_2$ be a morphism of finite-dimensional Lie algebras over a field of characteristic zero. Our problem is this: given a finite-dimensional L_1-module, V say, when does V embed as a sub L_1-module of some finite-dimensional L_2-module? The problem clearly reduces to the case in which ϕ is injective. We provide here (Thm. 3.6) a solution in two separate cases: (i) under the assumption that ϕ maps the radical of L_1 into the radical of L_2, or (ii) under the assumption that L_1 is its own commutator ideal.

0. Introduction. A theorem of Bialynicki-Birula, Hochschild, and Mostow ([**1**, Thm. 1]) gives conditions for a finite-dimensional module for a subgroup of an algebraic group to embed as a submodule into a finite-dimensional module for the whole group. It is with a modification of this result that we obtain criteria for modules of Lie algebras.

Throughout this paper, k will denote a field of characteristic zero, and K will be an algebraic closure of k. For a Lie algebra L over k, $U(L)$ will denote the universal enveloping algebra of L; $H(L)$ will denote the Hopf algebra of representative functions associated with L. All of our Lie algebras, modules, and representations are taken to be *finite-dimensional* unless otherwise specified. We will regard a module for a Lie algebra L as also a left $U(L)$-module or as a right $H(L)$-comodule, and vice versa.

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1. Reduction of the problem to representative functions.

DEFINITION. Let ϕ: $H_1 \to H_2$ be a morphism of coalgebras over k. ϕ induces an H_2-comodule structure on any H_1-comodule ψ: $V \to V \otimes H_1$ by following up ψ with $(i \otimes \phi)$, where i is the identity map. We say that an H_2-comodule ξ: $U \to U \otimes H_2$ is *extendable* to H_1 if there is an H_1-comodule ψ: $V \to V \otimes H_1$ and a linear injection j: $U \hookrightarrow V$ such that $(j \otimes i) \circ \xi = (i \otimes \phi) \circ \psi \circ j$.