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ANALYTICITY AND SPECTRAL DECOMPOSITIONS OF L^p FOR COMPACT ABELIAN GROUPS

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Let Γ be a dense subgroup of the real line $\mathbf{R}$. Endow Γ with the discrete topology, and let K be the dual group of Γ. Helson's classic theory uses the spectral representation in Stone's Theorem for unitary groups to establish and implement a one-to-one correspondence Φ_2 between the cocycles on K and the normalized simply invariant subspaces of $L^2(K)$. Using our recent extension of Stone's Theorem to UMD spaces, we generalize Helson's theory to $L^p(K)$, $1 < p < \infty$, by producing spectral decompositions of $L^p(K)$ which provide a correspondence analogous to Φ_2. In particular this approach shows that every normalized simply invariant subspace of $L^p(K)$ is the range of a bounded idempotent. However, unlike the situation in the L^2-setting, our spectral decompositions do not stem from a projection-valued measure. Instead they owe their origins to the Hilbert transform of $L^p(\mathbf{R})$. In the context of abstract UMD spaces, we develop the relationships between holomorphic semigroup extensions and the spectral decompositions of bounded one-parameter groups. The results are then applied to describe, in terms of generalized analyticity, the normalized simply invariant subspaces of $L^p(K)$.

More specifically, throughout what follows K will be a compact abelian group other than $\{0\}$ or the unit circle $\mathbf{T}$ such that the dual group of K is archimedean ordered. Equivalently, we shall require that K is the dual group of Γ, where Γ arises as a dense subgroup of the real line $\mathbf{R}$, and Γ is then endowed with the natural order of $\mathbf{R}$ and the discrete topology. For each $\lambda \in \Gamma$ we denote by χ_λ the corresponding character on K (evaluation at λ), and for each $t \in \mathbf{R}$ we let e_t be the element of K defined by $e_t(\lambda) = \exp(it\lambda)$ for all $\lambda \in \Gamma$. As is well-known, $t \to e_t$ is a continuous isomorphism of $\mathbf{R}$ onto a dense subgroup of K. For $1 < p < \infty$ we follow Helson in defining a *simply invariant subspace* of $L^p(K)$ to be a closed subspace M of $L^p(K)$ such that $\chi_\lambda M \subseteq M$ for all $\lambda > 0$, but for some $\alpha < 0$, $\chi_\alpha M$ is not a subset of M. A simply invariant subspace M is said to be *normalized* provided $M = \bigcap\{\chi_\lambda M \colon \lambda \in \Gamma,\ \lambda < 0\}$. The set of all normalized simply invariant subspaces of $L^p(K)$ will be denoted by $\mathscr{S}_p$. A *cocycle* on K is a Borel measurable function $A\colon \mathbf{R} \times K \to \mathbf{T}$ such that

$$A(t+u, x) = A(t, x)A(u, x+e_t), \quad \text{for } t \in \mathbf{R},\ u \in \mathbf{R},\ x \in K.$$