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NON-EXISTENCE OF CERTAIN CLOSED COMPLEX GEODESICS IN THE MODULI SPACE OF CURVES

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We prove that most compact totally geodesic curves in the Siegel moduli space A_g of g-dimensional principally polarized abelian varieties cannot lie in the image of the period mapping of the moduli space M_g of smooth curves of genus g. The meaning of "most" is in terms of the holomorphic sectional curvature of Siegel space—see the precise statement below.

The reason for studying this question is that it gives some idea of the differential geometry of the period mapping $M_g \to A_g$. This mapping is presumably rather curved, i.e., the image of M_g in A_g is curved relative to the locally symmetric geometry of A_g. The best way to make this precise would be to compute the second fundamental form of the period mapping. This could be an involved computation, perhaps not immediately interpretable in geometric terms. Hence we prefer first to take a more elementary approach and ask if the image of M_g contains any straight lines of the symmetric geometry, i.e., any complex totally geodesic curves. The easiest question to decide, and the only one studied here, is whether any closed complex geodesics lie in the image. The question of geodesics of finite area is quite interesting, but more difficult.

We study this question by applying the Miyaoka inequality [**5**] to the complex surface induced by the curve in M_g. It gives that the image of the classifying mapping for the cohomology of the fibers (period mapping) has area less than $1/3$ of the expected maximum for the area of a mapping into the period space. This strongly suggests curvature properties of the period mapping.

To show that the above restrictions are not vacuous, we point out in §3 that closed geodesics violating the restrictions do exist in A_g. These are constructed from classical examples of Hilbert modular surfaces in A_2. We also remark that, for $g \geq 3$, M_g has plenty of compact curves. Examples starting in $g = 6$ are explicitly constructed in [**1, 4**].

A final algebro-geometric remark is that complex geodesics in A_g are related to reducibility of the monodromy representation. Namely a complex geodesic of curvature $-1/l$ (cf. §1) parametrizes a family of abelian