PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 1, 1987

# PSEUDOGROUPS OF $C^1$ PIECEWISE PROJECTIVE HOMEOMORPHISMS

PETER GREENBERG

**The group $\mathrm{PSL}_2\mathbf{R}$ acts transitively on the circle $S^1 = \mathbf{R} \cup \infty$, by linear fractional transformations. A homeomorphism $g: U \to V$ between open subsets of R is called $C^1$, *piecewise projective* if $g$ is $C^1$, and if there is some locally finite subset $S$ of $U$ such that, on each component of $U - S$, $g$ agrees with some element of $\mathrm{PSL}_2\mathbf{R}$. Let $\Gamma_{\mathbf{R}}$ be the pseudogroup of such homeomorphisms. We show that the Haefliger classifying space $B\Gamma_{\mathbf{R}}$ is simply connected, and that there is a homology isomorphism $i$: $B\widetilde{\mathrm{PSL}_2\mathbf{R}} \to B\Gamma_{\mathbf{R}}$. ($\widetilde{\mathrm{PSL}_2\mathbf{R}}$ is the universal cover of $\mathrm{PSL}_2\mathbf{R}$, considered as a discrete group.) As a consequence, the classifying space of the discrete group of compactly supported, $C^1$ piecewise projective homeomorphisms of R is a "homology loop space" of $B\widetilde{\mathrm{PSL}_2\mathbf{R}}$.**

**1.1. Introduction.** More generally, let $F \subset \mathbf{R}$ be a subfield of $\mathbf{R}$. $\mathrm{PSL}_2F$ acts on the circle $\mathbf{R} \cup \infty$. The orbit of $1 \in F$ is $F \cup \infty$.

1.2. DEFINITION. $\Gamma_F$ is the pseudogroup of $C^1$ homeomorphisms $g$: $U \to V$ between open subsets of $\mathbf{R}$, so that there is some locally finite subset $S$ of $U \cap (F \cup \infty)$ such that, on each connected component of $U - S$, $g$ agrees with some element of $\mathrm{PSL}_2F$.

The set of restrictions of elements of $\mathrm{PSL}_2F$ to open subsets of $\mathbf{R}$ forms a subpseudogroup of $\Gamma_F$ whose classifying space, the total space of the circle bundle over $B\mathrm{PSL}_2F$, is homotopy equivalent to $B\widetilde{\mathrm{PSL}_2F}$, where $\widetilde{\mathrm{PSL}_2F}$ is defined as the pullback

$$\begin{array}{ccc} \widetilde{\mathrm{PSL}_2F} & \to & \widetilde{\mathrm{PSL}_2\mathbf{R}} \\ \downarrow & & \downarrow \\ \mathrm{PSL}_2F & \to & \mathrm{PSL}_2\mathbf{R} \end{array}$$

Therefore, there is an inclusion map $i$: $B\widetilde{\mathrm{PSL}_2F} \to B\Gamma_F$.

1.3. THEOREM. *$\pi_1 B\Gamma_F = 0$, and $i$ is a homology equivalence.*

1.4. DEFINITION. The *group of compactly supported $\Gamma_F$ homeomorphisms*, denoted $K_F$, is the group of elements of $\Gamma_F$ which are compactly supported homeomorphisms of the line $\mathbf{R}$.