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QUOTIENTS OF THE COMPLEX BALL BY DISCRETE GROUPS

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In this paper we systematically study varieties $Q(\underline{\mu})$, which are compactifications of the space Q of distinct points in $(\mathbf{P}^1)^r$ given by a sequence of "weights" $\underline{\mu}$, and which for certain $\underline{\mu}$ are also compactification of the quotient of the complex r-ball by discrete subgroups $\Gamma(\underline{\mu})$ of $\mathrm{PU}(r, 1)$, as discovered by Deligne and Mostow.

We obtain a wealth of topological information about the spaces $Q(\underline{\mu})$ and their desingularizations $Q^*(\underline{\mu})$. In some cases we can completely describe them. Otherwise, we obtain computations of Betti numbers and Hodge numbers. As applications we determine the L^2-cohomology and in many cases the (ordinary) rational cohomology of the groups $\Gamma(\underline{\mu})$.

0. Introduction. In this paper we study a family of algebraic varieties which arise in two ways. The first is as quotients of the ball in $\mathbf{C}^r$ by discrete subgroups of $\mathrm{PU}(r, 1)$, and the second as various compactifications of the configuration space Q of r distinct points in $\mathbf{P}^1$.

They were first discovered by Deligne and Mostow ([**DM**], [**M**]) where they arose through the investigation of generalized hypergeometric functions. We briefly recapitulate their work in §2 below.

We wish to study those varieties systematically. In this connection, we find that the second viewpoint, in terms of Mumford's geometric invariant theory [**Mu**], is more useful. Let $N = r + 3$ and let $\underline{\mu} = (\mu_1, \ldots, \mu_N)$ be a sequence of positive integers (which we call "weights"). Associated to $\underline{\mu}$ is a line bundle over $(\mathbf{P}^1)^N$, and hence an r-dimensional projective variety $Q(\underline{\mu})$ obtained by taking the semistable points with respect to the linear action of PGL_2 and then forming the quotient space in the sense of geometric invariant theory (see §6). $Q(\underline{\mu})$ is a compactification of Q, and the varieties of Deligne and Mostow arise as $Q(\underline{\mu})$ for $\underline{\mu}$ satisfying certain arithmetical conditions.

These varieties $Q(\underline{\mu})$ are always rational (1.11). When $r = 2$, and when $r = 3$ and $Q(\underline{\mu})$ is nonsingular, they can be completely described (4.1). (When $r = 2$ the possibilities for $Q(\underline{\mu})$ are $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ with k points blown up, $0 \leq k \leq 4$. When $r = 3$ the possibilities are more complicated.) In this case our work also determines the rational cohomology of the associated discrete subgroup $\Gamma(\underline{\mu})$ of $\mathrm{PU}(r, 1)$.