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THE INDETERMINATE RATE PROBLEM FOR BIRTH-DEATH PROCESSES

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A birth-death process is completely determined by its set of rates if and only if this set satisfies a certain condition C, say. If for a set of rates R the condition C is not fulfilled, then the problem arises of characterizing all birth-death processes which have rate set R (the indeterminate rate problem associated with R). We show that the characterization may be effected by means of the decay parameter, and we determine the set of possible values for the decay parameter in terms of R. A fundamental role in our analysis is played by a duality concept for rate sets, which, if the pertinent rate sets satisfy C, obviously leads to a duality concept for birth-death processes. The latter can be stated in a form which suggests the possibility of extension in the context of indeterminate rate problems. This, however, is shown to be only partially true.

1. Introduction. Let $\mathscr{P} = \{p_{ij}(t) \mid i, j \in E', 0 \le t < \infty\}$ be the set of transition probability functions of a standard, time-homogeneous Markov process $X(t)$ on the state space $E' \equiv \{-1, 0, 1, \dots\}$. That is,

$$p_{ij}(t) \equiv \Pr\{X(s+t) = j \mid X(s) = i\},$$

and

$$p_{ij}(t) \ge 0, \tag{1.1}$$

$$\sum_{k \in E'} p_{ik}(t) \le 1, \tag{1.2}$$

$$\sum_{k \in E'} p_{ik}(s) p_{kj}(t) = p_{ij}(s+t), \tag{1.3}$$

$$\lim_{t \downarrow 0} p_{ij}(t) = p_{ij}(0) = \delta_{ij}, \tag{1.4}$$

$(i, j \in E';\ s, t \ge 0)$. $\mathscr{P}$ is said to represent a birth-death process if the Q-matrix $Q = (q_{ij})$ of $\mathscr{P}$, defined by

$$q_{ij} \equiv \lim_{t \downarrow 0} t^{-1}\left(p_{ij}(t) - \delta_{ij}\right),$$

satisfies $q_{ij} = 0$ if $i = -1$ or $|i - j| > 1$, $0 < q_{ij} < \infty$ if $i \ge 0$ and $|i - j| = 1$ (with the exception $0 \le q_{0,-1} < \infty$), and

$$q_{ii} = -(q_{i,i-1} + q_{i,i+1}), \qquad i \ge 0.$$