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LIPSCHITZ CONVERGENCE OF RIEMANNIAN MANIFOLDS

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Let $\mathscr{C} \equiv \mathscr{C}(n, \Lambda, \delta_0, V_0)$ be the set of all connected compact C^∞ n-dimensional Riemannian manifolds with |sectional curvature| $< \Lambda^2$, diameter $< \delta_0$, and volume $> V_0$. The main result of this paper is that this class $\mathscr{C}$ has certain compactness, or more precisely, precompactness properties. The class $\mathscr{C}$ consists of only finitely many diffeomorphism classes so the precompactness properties can be thought of as dealing with the set of metrics satisfying the class $\mathscr{C}$ requirements on a fixed differentiable manifold. The main theorem of this paper is then that a sequence of such metrics always has a subsequence which, after application of suitable diffeomorphisms of M, converges to a limit metric. The regularity of the limit can be taken to be $C^{1,\alpha}$, for all α with $0 < \alpha < 1$ and the convergence to be in the $C^{1,\alpha}$ norm.

In [**12**], M. Gromov stated, in terminology that will be explained in a moment, a striking convergence theorem concerning the class $\mathscr{C}$ defined above (Theorems 8.25 and 8.28 of [**12**]):

(∗) Given a sequence $\{M^l\colon l = 1, 2, 3, \dots\}$ in $\mathscr{C}$, there exists a subsequence $\{M^k\}$ and a $D^{1,1}$-Riemannian manifold M such that $\{M^k\}$ converges to M in the Lipschitz distance.

Here a $C^{1,1}$-manifold is by definition a C^1-manifold with coordinate transition functions having Lipschitz continuous first derivatives (note that the notion of Lipschitz continuity used here is that of Euclidean space to Euclidean space, so no choice of metric is involved). A $D^{1,1}$-Riemannian manifold M is a $C^{1,1}$-manifold with a continuous Riemannian metric having the additional property that, for each point x in M, the distance function ρ_x determined by the metric is C^1 (within the cut locus) and that the derivatives of ρ_x are Lipschitz continuous with a Lipschitz constant that is independent of the point x. The Lipschitz distance $d_L(M_1, M_2)$ between two homeomorphic compact metric spaces M_1, M_2 is defined to be the infimum of

$$|\log \operatorname{dil}(f)| + |\log \operatorname{dil}(f^{-1})|$$