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GRUNSKY INEQUALITIES FOR UNIVALENT FUNCTIONS WITH PRESCRIBED HAYMAN INDEX

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The Grunsky inequalities in their standard formulation are a generalization of the area principle. Our purpose is to apply a variational method to obtain a stronger system of inequalities which involves both the logarithmic coefficients and the Hayman index of a univalent function f in the usual class S. One immediate consequence is the well-known inequality of Bazilevich on logarithmic coefficients. Another application gives a sharpened form of the Goluzin inequalities on the values of f at prescribed points of the disk.

1. Main results. The class S consists of all functions $f(z) = z + a_2 z^2 + \cdots$ analytic and univalent in the unit disk $\mathbf{D}$. Closely related is the class Σ of all functions

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

analytic and univalent in the exterior $\Delta = \mathbf{C} - \overline{\mathbf{D}}$ of the disk. Given $g \in \Sigma$ we construct the double power series

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} z^{-n} \zeta^{-m}, \qquad z, \zeta \in \Delta.$$

Then $d_{mn} = d_{nm}$ and the *Grunsky inequalities* ([**6**]; see [**3**], Chapter 4) take the form

$$\left| \sum_{n=1}^{N} \sum_{m=1}^{N} d_{nm} \lambda_n \lambda_m \right| \leq \sum_{n=1}^{N} \frac{1}{n} |\lambda_n|^2, \qquad \lambda_n \in \mathbf{C}.$$

Now let $f \in S$ and consider the analogous series

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} z^n \zeta^m \tag{1}$$

for z, $\zeta \in \mathbf{D}$. Note that $c_{mn} = c_{nm}$ and $c_{00} = 0$. If $\zeta = 0$ the series reduces to

$$\log \frac{f(z)}{z} = -\sum_{n=0}^{\infty} c_{n0} z^n.$$

The *inversion* of f is the function $g \in \Sigma$ defined by $g(1/z) = 1/f(z)$. Thus

$$\frac{g(1/z) - g(1/\zeta)}{1/z - 1/\zeta} = \frac{f(z) - f(\zeta)}{z - \zeta} \cdot \frac{z\zeta}{f(z) f(\zeta)}.$$