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ON COVERING OF REAL LINE BY NULL SETS

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In this note it is proved that the least cardinal κ such that R cannot be covered by κ many null sets cannot have countable cofinality, provided 2^ω-scale exists and 2^ω is regular cardinal. Using the same assumption a combinatorial characterisation of this cardinal is also found.

Let κ_m be the least cardinal κ such that the real line can be covered by κ many null sets.

The goal of this paper is to give a combinatorial description of κ_m. This problem was stated by Arnold Miller in **[Mi 1]**. We also aim to find a solution to the question asked by David Fremlin whether $cf(\kappa_m)$ can be equal ω.

We are able to answer the above questions using the additional assumption of the existence of 2^ω-scale in ω^ω.

It is well known that the cardinal κ_m is the same in **R**, as in 2^ω or any uncountable Polish space with totally σ-finite continuous measure. Thus without loss of generality we can work in the space 2^ω with the Lebesgue measure.

We shall use standard notation. ω denotes the set of natural numbers. For $n, k < \omega$ the interval $[n, k)$ is the set $\{i < \omega\colon n \leq i < k\}$. For $n < \omega$ 2^n (2^ω) is either the set of all 0-1 sequences of length n (ω) or the cardinal 2^n (2^ω)—depending on the context. For any set X $|X|$ denotes the cardinality of X. For any two finite sequences s, t $s \wedge t$ denotes concatenation of them. Finally, quantifiers $\forall^\infty$ and $\exists^\infty$ abbreviate "for all except finitely many" and "there exist infinitely many" respectively.

1. In this first section we introduce the notation of small sets and we prove basic properties of these sets.

We start with the following well-known lemma.

LEMMA 1.1. *Suppose $H \subseteq 2^\omega$ is a null set. Then there exists a sequence $J_n \subseteq 2^n$ for $n < \omega$ such that $\sum_{n=1}^\infty |J_n| 2^{-n} < \infty$ and $H \subseteq \{x \in 2^\omega\colon \exists^\infty n$ $x \restriction n \in J_n\}$.* $\square$

For the proof of Lemma 1.1 see **[0]**.