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APPROXIMATION PROPERTIES FOR SOME NON-NOETHERIAN LOCAL RINGS

H. SCHOUTENS

In this paper we study Artin approximation in power series rings in several variables over complete rank-one valuation rings. In particular we prove that the completion of the algebraic elements has the approximation property over the ring of algebraic power series.

Moreover, for an important class of complete rank-one valuation rings, e.g. the ring of complex p-adic integers, we prove that the ring of algebraic power series is equal to the henselisation of the polynomial ring and that each algebraic power series has coefficients lying in a finitely generated R-algebra, where R is discrete valuation rings.

1.1. Let $R \subset \overline{R}$ be a pair of rings (always commutative, with unity). We'll consider topologies on $\overline{R}$ which stem from a filtration of ideals $\mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ which tends to zero, i.e. $\bigcap_{n=0}^{\infty} \mathfrak{a}_n = 0$. Two examples we will use are given by

(1) an ideal $\mathfrak{a}$ of $\overline{R}$ with $\bigcap_{n=0}^{\infty} \mathfrak{a}^n = 0$ and where $\mathfrak{a}_n = \mathfrak{a}^n$; thus we get the $\mathfrak{a}$-adic topology,

(2) a rank-one valuation on $\overline{R}$ (i.e. a valuation with value group in the positive real numbers) and $\mathfrak{a}_n = \{x \in \overline{R} \mid v(x) \geq n\}$.

DEFINITION. $\overline{R}/R$ has *A. P.* (*(Artin)-approximation property*) when the following holds: For every system of polynomial equations $f = 0$ over R, i.e. $f = (f_1, \ldots, f_q)$ with $f_i \in R[Y_1, \ldots, Y_N]$ which has a solution $\bar{y}$ in $\overline{R}^N$, we can find for each n in $\mathbf{N}$ a solution y in R^N such that $\bar{y} \equiv y \bmod \mathfrak{a}_n$.

REMARK. Often, one can express congruence conditions such as "$\bar{y} \equiv y \bmod \mathfrak{a}_n$" appearing in the definition, by polynomial conditions. More explicitly, let I be a finitely generated ideal of R, such that R is dense in $\overline{R}$ with respect to the $I\overline{R}$-adic topology. Let $f \in R[Y]^q$, $Y = (Y_1, \ldots, Y_N)$ and $\bar{y} \in \overline{R}^N$ s. t. $f(\bar{y}) = 0$. We look for a solution $y \in R^N$ such that $y \equiv \bar{y} \bmod I^m$, for a chosen $m \in \mathbf{N}$. Let $I^m = (q_1, \ldots, q_s)$ with $q_i \in R$, and since $R \subset \overline{R}$ is dense, we can find $\hat{y} \in R^N$, $\bar{\alpha}_1, \ldots, \bar{\alpha}_s \in \overline{R}^N$ s.t. $\bar{y} = \hat{y} + \bar{\alpha}_1 q_1 + \cdots + \bar{\alpha}_s q_s$.