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ALGEBRAICALLY DEFINED SUBSPACES IN THE COHOMOLOGY OF A KUGA FIBER VARIETY

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Let G be a semisimple algebraic group of hermitian type defined over Q, let $\mathfrak{X} \simeq G_{\mathbf{R}}/K$, where $K \subset G$ is a maximal compact subgroup, be the symmetric domain associated to G, let Γ be an arithmetic subgroup of G, let (π, E) be a finite-dimensional representation of G defined over Q, let $\mathscr{U} := \Gamma \backslash \mathfrak{X}$, and let $\mathscr{E}$ be the locally constant sheaf over $\mathscr{U}$ associated to (π, E). Then under certain conditions on G, Γ and (π, E), the quotient $\mathscr{U}$ is a complex projective variety and there exists a Kuga fiber variety $\mathscr{V}$, i.e., a complex projective variety with the structure of an analytic family of abelian varieties parametrized by $\mathscr{U}$, such that $H^a(\mathscr{U}; \mathscr{E})$ may be identified with a subspace of $H^*(\mathscr{V}; \mathbf{Q})$. The purpose of this paper is to show that for a certain class of nontrivial (π, E) the subspace of $H^*(\mathscr{V}; \mathbf{Q})$ with which $H^a(\mathscr{U}; \mathscr{E})$ is identified is algebraically defined, or in other words that this subspace is contained in $H^r(\mathscr{V}; \mathbf{Q})$ for some r and a projection from $H^r(\mathscr{V}; \mathbf{Q})$ to it is induced by an algebraic class in $H^*(\mathscr{V} \times \mathscr{V}; \mathbf{Q})$. In particular, since the projection of an algebraic class in $H^r(\mathscr{V}; \mathbf{Q})$ is again an algebraic class, this paper provides an answer to the question of how to define algebraic classes in $H^a(\mathscr{U}; \mathscr{E})$ for some nontrivial local coefficient systems $\mathscr{E}$.

Introduction. Let G be a semisimple algebraic group defined over $\mathbf{Q}$, of hermitian type and $\mathbf{Q}$-rank zero, and let Γ be a torsion-free arithmetic subgroup of G which is Zariski dense in G even when $G_{\mathbf{R}}$ has compact factors [1] and contained in the identity component of $G_{\mathbf{R}}$. Further, let $\mathfrak{X} \simeq G_{\mathbf{R}}/K$, for some maximal compact $K \subset G_{\mathbf{R}}$, be the symmetric domain associated to G, and let (σ, W, β) be a symplectic representation of G defined over $\mathbf{Q}$. In 1963 Kuga [12] showed how to associate to $(G, \Gamma, \mathfrak{X}, (\sigma, W, \beta))$ together with some additional data an analytic family $\mathscr{V}$ of abelian varieties parametrized by the locally symmetric variety $\mathscr{U} := \Gamma \backslash \mathfrak{X}$ such that if (σ, W, β) satisfies a certain analyticity condition (condition (H_1), cf. paragraph 1A below) then $\mathscr{V}$ is a complex projective variety. Such a variety $\mathscr{V}$ is called a Kuga fiber variety; cf. paragraph 1C below for more details.

One of Kuga's many motivations for studying Kuga fiber varieties was the observation that they provide a natural algebraic-geometric realization for the cohomology of arithmetic groups with nontrivial coefficients. That is, suppose (π, E) is a representation of G defined over $\mathbf{Q}$