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POLYNOMIAL EQUATIONS OF IMMERSED SURFACES

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If V is a nonsingular real algebraic set we say $H_i(V; \mathbf{Z}_2)$ is algebraic if it is generated by nonsingular algebraic subsets of V.

Let V^3 be a 3-dimensional nonsingular real algebraic set. Then, we prove that any immersed surface in V^3 can be isotoped to an algebraic subset if and only if $H_i(V: \mathbf{Z}_2)$ $i = 1, 2$ are algebraic. This isotopy above carries the natural stratification of the immersed surface to the algebraic stratification of the algebraic set. Along the way we prove that if V is any nonsingular algebraic set then any simple closed curve in V is ε-isotopic to a nonsingular algebraic curve if and only if $H_1(V: \mathbf{Z}_2)$ is algebraic.

Let V^3 be a 3-dimensional nonsingular real algebraic set. We call a homology group of V algebraic if it is generated by nonsingular algebraic subsets. In this paper we prove:

THEOREM. *The following are equivalent:*

(a) *If $f: M^2 \looparrowright V^3$ is any immersion of a closed smooth surface in general position, then $f(M^2)$ is isotopic to an algebraic subset Z of V^3 by an arbitrarily small isotopy. This isotopy carries the natural stratification of $f(M^2)$ to the algebraic stratification of Z.*

(b) *$H_1(V; \mathbf{Z}_2)$ and $H_2(V; \mathbf{Z}_2)$ are algebraic.*

To be more precise for $i = 1, 2$ let $AH_i(V^3; \mathbf{Z}_2)$ be the subgroup of $H_i(V^3; \mathbf{Z}_2)$ generated by nonsingular algebraic subsets. Then $H_i(V; \mathbf{Z}_2)$ is algebraic if it is equal to $AH_i(V; \mathbf{Z}_2)$. In particular zero homology groups are algebraic. We will refer to elements of $AH_i(V^3; \mathbf{Z}_2)$ as algebraic homology classes. This definition is consistent with the conventions of [**AK**$_1$].

In case f is an imbedding this theorem reduces to a special case of Proposition 1 below, which is Theorem 4.1 and Remark 4.2 of [**AK**$_1$]. Recall, if W^n is a nonsingular algebraic set of dimension n, then $AH_{n-1}(W; \mathbf{Z}_2)$ is the subgroup of $H_{n-1}(W; \mathbf{Z}_2)$ generated by nonsingular algebraic subsets. Also if $M \subset W$ is a closed submanifold, denote the