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FIRST INTEGRALS FOR A DIRECTION FIELD ON A SIMPLY CONNECTED PLANE DOMAIN

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Let $\mathscr{D}$ be a simply connected plane domain and let r be a positive integer or ∞. By a C^r direction field Φ on $\mathscr{D}$ we mean C^r mapping Φ: $\mathscr{D} \to G_{2,1}$, the projective line consisting of lines through the origin, 0, in the plane. A C^r first integral of Φ is a C^r function f: $\mathscr{D} \to \mathbf{R}$ such that each level set of f has no interior and is a union of members of the family, $\mathscr{F}$, of maximal integral curves of Φ. We show, in general, that first integrals do not exist and then give a necessary and sufficient condition for a C^r first integral to exist. When Φ has a first integral we also show that there exists a local diffeomorphism μ: $\mathscr{D} \to \mathbf{R}^2$ such that Φ is mapped by μ into the (constant) vertical direction field on $\mu(\mathscr{D})$.

Introduction. W. Kaplan [**4**] has shown that a C^r direction field Φ has a C^0 first integral; more precisely, he shows there exists a continuous function f: $\mathscr{D} \to \mathbf{R}$ such that

(1) for every $c \in \mathbf{R}$, $f^{-1}(c)$ consists of at most countably many curves of $\mathscr{F}$, and

(2) in every neighborhood of a point $P_0 \in \mathscr{D}$ there are points P for which $f(P) > f(P_0)$ and points P for which $f(P) < f(P_0)$.

E. Kamke [**3**] and R. Finn [**1**] showed the existence of a C^r first integral for a simply connected relatively compact domain $\mathscr{D}$, assuming, roughly, that Φ has a C^r extension to the closure $\overline{\mathscr{D}}$; the Kamke Φ should be C^r on a domain containing $\overline{\mathscr{D}}$, while for Finn the boundary of $\mathscr{D}$ should be C^r. Finn, in fact, establishes necessary and sufficient conditions for the existence of first integrals in multiply connected domains and obtains conditions that apply to singular situations.

Since, as is well known, all simply connected domains in the plane are diffeomorphic, we will take $\mathscr{D}$ to be the plane, $\mathbf{R}^2$, throughout. However, for specific applications and examples it is often convenient to use other models. Thus we are led to the following question. If Φ is C^r on $\mathbf{R}^2$, does there exist a globally defined C^r first integral of Φ without imposing additional restrictions on Φ? In §1 we show that additional conditions must be imposed. This is a consequence of the following theorem.