PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 2, 1988

LOEWY SERIES AND SIMPLE PROJECTIVE MODULES IN THE CATEGORY $\mathcal{O}_S$

RONALD S. IRVING AND BRAD SHELTON

Results are obtained on the Loewy length and Loewy series of generalized Verma modules and projective modules in certain categories $\mathcal{O}_S$ of modules over a complex, semisimple Lie algebra. The results obtained rely on a study of the behavior of Loewy series under translation functors and on the existence of simple projective modules in suitable blocks of $\mathcal{O}_S$. An example is given of two generalized Verma modules such that the space of $\mathcal{O}_S$-homomorphisms from the first to the second is two-dimensional.

1. Introduction.

1.1. In this paper we study the Loewy series of generalized Verma modules and self-dual projectives in the category $\mathcal{O}_S$ associated to a complex semisimple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}_S$. The principal theme is the translation of data from a block of $\mathcal{O}_S$ associated to a non-regular weight to the blocks arising from other, possibly regular, weights, especially in case the first block contains a simple, projective module. In particular, we find that $\mathcal{O}_S$ contains simple projectives for any choice of parabolic subalgebra if $\mathfrak{g}$ is of type A_n, and thereby obtain precise formulas for Loewy length in $\mathcal{O}_S$. For other Lie algebras, the presence of a simple projective depends on the choice of $\mathfrak{p}_S$. We consider some cases where simple projectives do exist, and study in detail the smallest example in which a simple projective fails to exist: $\mathfrak{g}$ is of type D_4 and $\mathfrak{p}_S$ is a minimal parabolic. For this example we find a generalized Verma module whose socle is a direct sum of two isomorphic copies of a simple module. Thus, we obtain a pair of generalized Verma modules whose space of homomorphisms is two-dimensional.

1.2. To describe our results in more detail, recalling earlier related work along the way, we need to introduce some notation to be used throughout the paper. Any unexplained terminology can be found in [8], [9]. However, some of the notation below differs from that in [8], [9] because of the convention used here for highest weights of Verma modules. We fix a complex simple Lie algebra $\mathfrak{g}$ with Cartan