PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 2, 1988

FUNDAMENTAL DOMAINS FOR THE GENERAL LINEAR GROUP

DOUGLAS GRENIER

Historically the most familiar fundamental domain for $P_n/\mathrm{GL}_n(\mathbf{Z})$ has been that of Minkowski. This paper develops a new fundamental domain more suited to applications in number theory. It is shown that these domains can be determined explicitly for given n and this is done for $n = 3, 4, 5, 6$. A reduction algorithm for an arbitrary element of P_n is also determined.

1. Introduction. Throughout this paper, let P_n denote the space of positive definite, symmetric, real $n \times n$ matrices. The identity matrix will always be denoted by I or I_n where necessary to avoid ambiguity. If $G = \mathrm{GL}_n(\mathbf{R})$, the general linear group over $\mathbf{R}$, and K is the subgroup of G of orthogonal matrices, P_n can be identified with $K\backslash G$ as follows:

$$\begin{aligned} K\backslash G &\to P_n \\ Kg &\to {}^T g g \end{aligned}$$

where ${}^T g$ denotes the transpose of the matrix g. We can define an action of the group G on P_n by ${}^T gYg$ for $g \in G$ and $Y \in P_n$. We will use the notation $Y[g] = {}^T gYg$. Now, as $\mathrm{GL}_n(\mathbf{Z})$ is a discrete subgroup of G, and so acts discontinuously on P_n, we can define a fundamental domain $P_n/\mathrm{GL}_n(\mathbf{Z})$. If Γ is any discrete subgroup of G, then a fundamental domain for P_n/Γ is a subset of P_n satisfying two conditions:

(1) The union of the images under the action of Γ covers P_n, i.e.,

$$\bigcup_{\gamma \in \Gamma} \gamma(P_n/\Gamma) = P_n.$$

(2) If Y and $Y[g]$, $g \in \Gamma$, are both in the fundamental domain, then Y and $Y[g]$ are on the boundary of the fundamental domain or $g = I$. From here on, unless otherwise noted, Γ will always be $\mathrm{GL}_n(\mathbf{Z})$.

Historically, the standard fundamental domain for P_n/Γ has been that of Minkowski, [9], here denoted M_n. M_n is defined as follows:

$$M_n = \{Y \in P_n \mid Y[a] \geq y_{ii} \text{ if } a \in \mathbf{Z}^n, \text{g.c.d.}(a_i, \dots, a_n) = 1; \quad y_{i,i+1} \geq 0 \text{ for } i = 1, \dots, n-1\}.$$