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A SEMILINEAR WAVE EQUATION WITH NONMONOTONE NONLINEARITY

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We prove that a semilinear wave equation in which the range of the derivative of the nonlinearity includes an eigenvalue of infinite multiplicity has a solution. The solution is obtained through an iteration scheme which provides a priori estimates.

1. Introduction. Here we study the nonlinear wave equation

$$u_{tt} - u_{xx} + \lambda u = cq(x,t) + r(x,t) + g(u), \quad (x,t) \in [0,\pi] \times \mathbf{R}, \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0, \qquad u(x,t) = u(x,t+2\pi), \quad (x,t) \in [0,\pi] \times \mathbf{R}, \tag{1.2}$$

where $\lambda \in \mathbf{R} - \{k^2 - j^2 : k = 1,2,3,\dots, j = 0,1,2,\dots\}$ and g is a function of class C^1 such that

$$\lim_{|u|\to\infty} g'(u) = 0. \tag{1.3}$$

A main difficulty in studying (1.1)–(1.2) arises when $-\lambda \in g'(\mathbf{R})$. This causes compactness arguments to fail because 0 is an eigenvalue of $u_{tt} - u_{xx}$, (1.2) of infinite multiplicity. Recent studies on (1.1)–(1.2) either: (i) assume that $g(u) - \lambda u$ is monotone (see **[B-N]**, **[R]**, **[W]**), or (ii) assume enough symmetry on g, q, and r so that the kernel of $u_{tt} - u_{xx}$, (1.2) reduces to $\{0\}$ (see **[Co]**), or (iii) use dichotomy on whether the Palais-Smale condition holds proving existence for values of $cq + r$ which cannot be given explictly (see **[H]**, **[W]**). Our main result (Theorem A below) does not fall in any of the above three classes.

Let $\Omega = [0,\pi] \times [0,2\pi]$. Let H^1, L^2, and L^∞ denote the Sobolev spaces $H^1(\Omega)$, $L^2(\Omega)$, and $L^\infty(\Omega)$ respectively. We let $\| \ \|_1$, $\| \ \|$, and $\| \ \|_\infty$ denote the norms in H^1, L^2, and L^∞ respectively. Let

$$N = \left\{ u \in L^2 : u = \sum_{k=1}^{\infty} (a_k \sin(kx)\sin(kt) + b_k \sin(kx)\cos(kt)) \right\}.$$

Let $N^\perp \subsetneqq L^2$ denote the orthogonal complement to N in L^2. Let P denote the orthogonal projection onto N and Q the othogonal projection