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HÖLDER CONTINUITY OF THE GRADIENT AT A CORNER FOR THE CAPILLARY PROBLEM AND RELATED RESULTS

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It is well-known that solutions of the capillary problem are smooth when the boundary and contact angle are smooth. Using fairly deep methods which are specific to the capillary problem, Simon and Tam have proved the smoothness of the solution at a corner. Here the smoothness is considered in the context of general nonlinear boundary value problems. The primary tool is a maximum principle argument.

Let Ω be a bounded domain in $\mathbf{R}^2$ with unit inner normal γ, let κ and ϕ be positive constants with $\phi < \pi$, and consider the problem

$$\begin{gathered} \operatorname{div}((1+|Du|^2)^{-1/2}Du) = \kappa u \quad \text{in } \Omega, \\ (1+|Du|^2)^{-1/2}Du\cdot\gamma = \cos\phi \quad \text{on } \partial\Omega. \end{gathered} \tag{0.1}$$

When $\partial\Omega$ is sufficiently smooth, it is well-known that (0.1) has a unique, smooth solution. Specifically, $\partial\Omega \in C^{2,\alpha}$ implies $u \in C^{2,\varepsilon}(\overline{\Omega})$ for some $\varepsilon > 0$ by [7], [27] (in fact $\varepsilon = \alpha$ by [15, Lemma 2′]) while $\partial\Omega \in C^{1,1}$ implies $u \in C^{1,\beta}$ for any $\beta < 1$ by [7], [20], [27]. If ϕ is suitably restricted, (0.1) has a unique solution (in an appropriate weak sense) even for nonsmooth domains (see [5], [6]). Under various hypotheses, this solution may be unbounded [3] or bounded but discontinuous [12]. Our interest here is with circumstances under which u will be C^1: We assume that $\partial\Omega$ is the union of finitely many smooth curves which meet at an angle θ in the range $(0,\pi)$. If $\theta > |2\phi - \pi|$ (which is easily seen to be necessary for (0.1) to have a C^1 solution), Simon [25] has shown that $u \in C^1$. We improve this result by showing that $u \in C^{1,\varepsilon}$ for some computable ε and by considering more general differential equations and boundary conditions. Moreover our method rests on a simple application of the maximum principle, and we can prove regularity results in more than two dimensions. A related argument was used by Miersemann [23] (in two dimensions) when the quantity κu in (0.1) is replaced by a constant to prove $C^{1,\varepsilon}$ regularity in a corner. His method does not readily extend to (0.1) but ours includes his situation. The biggest differences