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CONTROLLED HOMOTOPY TOPOLOGICAL STRUCTURES

C. BRUCE HUGHES

Let $p : E \to B$ be a locally trivial fiber bundle between closed manifolds where $\dim E \geq 5$ and B has a handlebody decomposition. A *controlled homotopy topological structure* (or a *controlled structure*, for short) is a map $f : M \to E$ where M is a closed manifold of the same dimension as E and f is a $p^{-1}(\varepsilon)$-equivalence for every $\varepsilon > 0$ (see §2). It is the purpose of this paper to develop an obstruction theory which answers the question: *when is f homotopic to a homeomorphism, with arbitrarily small metric control measured in B?* This theory originated with an idea of W. C. Hsiang that a controlled structure gives rise to a cross-section of a certain bundle over B, associated to the Whitney sum of $p : E \to B$ and the tangent bundle of B.

1. Introduction. In §3 we define a semi-simplicial complex $\mathscr{S}(p : E \to B)$, called the *space of controlled structures on $p : E \to B$*. Roughly, an n-simplex of $\mathscr{S}(p : E \to B)$ is an n-parameter family of controlled structures on $p : E \to B$. The study of the homotopy relation in $\mathscr{S}(p : E \to B)$ was initiated in [$\mathbf{H_1}$, §8]. For example, if $f : M \to E$ is a controlled structure, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism for every $\varepsilon > 0$ if and only if $[f] = [\mathrm{id}]$ in $\pi_0\mathscr{S}(p : E \to B)$. The higher homotopy groups of $\mathscr{S}(p : E \to B)$ have analogous implications concerning parameterized versions of this problem (see §3). The main objective then is to understand the homotopy type of $\mathscr{S}(p : E \to B)$. This is accomplished as follows.

Let $\hat{p} : TB \oplus E \to B$ be the Whitney sum of the tangent bundle of B and E. This new bundle has fiber $\mathbf{R}^m \times F$ where $m = \dim B$ and F is the fiber of $p : E \to B$. In §5 we construct an associated bundle $\tilde{p} : \tilde{E} \to B$ with fiber $\mathscr{S}(\pi : \mathbf{R}^m \times F \to \mathbf{R}^m)$ where $\pi : \mathbf{R}^m \times F \to \mathbf{R}^m$ denotes projection. The main result of this paper is the following theorem.

THEOREM 1. *The space of controlled structures $\mathscr{S}(p : E \to B)$ is homotopy equivalent to the semi-simplicial complex of cross-sections of $\tilde{p} : \tilde{E} \to B$.*