PACIFIC JOURNAL OF MATHEMATICS
Vol. 133, No. 1, 1988

ON THE DIOPHANTINE EQUATION $1 = \sum 1/n_i + 1/\prod n_i$ AND A CLASS OF HOMOLOGICALLY TRIVIAL COMPLEX SURFACE SINGULARITIES

LAWRENCE BRENTON AND RICHARD HILL

Let $n_1, \ldots, n_N$ be integers ≥ 2, and let $x \in X$ be an isolated two-dimensional complex singularity whose dual intersection graph is a star with central weight 1 and with weights n_i on the arms. Then X is locally the cone on a homology 3-sphere if and only if $\sum n_i^{-1} + \prod n_i^{-1} = 1$. All such unit fraction expressions for 1 are given for $N \leq 7$, and properties of such sequences $\{n_i\}$ are discussed in general.

In this paper we will establish a correspondence between two-dimensional complex singularities whose local fundamental group is perfect and whose dual intersection graph is a star, and solutions in integers to the equation

$$1 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i}. \tag{1}$$

Next we will discuss techniques for finding solutions to (1). We have found all solutions (there are a total of 42) for $N \leq 7$, and many further examples for larger N. Our techniques involve both elementary number theoretic methods and computer-aided searches. These in turn give rise to several unanswered questions in the theory of Egyptian fractions, the most general being as follows (Professor Erdös offers \$100 for a solution): Let $n_0, n_1, \ldots, n_k$ be positive integers, relatively prime in pairs, with $n_i \geq 2$ for $i > 0$. Under what conditions do there exist integers $n_{k+1}, \ldots, n_N$, all ≥ 2, such that

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i}? \tag{2}$$

It might be remarked that no solution to (2) is known for $n_0 > 1$.

We wish to thank our colleagues Daniel Drucker, Judith Longyear, and Steven Williams for many useful conversations and suggestions. We also thank the referee for improving the exposition by calling our attention to some relevant work by other authors.