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ON DIVISORS OF SUMS OF INTEGERS, III

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In this paper we show that if $A_1, A_2, \dots, A_k$ are "dense" sets of integers, then there is a sum $a_1 + a_2 + \cdots + a_k$ with $a_1 \in A_1$, $a_2 \in A_2, \dots, a_k \in A_k$ that is divisible by a "small" prime.

1. Let $P(n)$ and $p(n)$ denote the greatest and smallest prime factor of n, respectively. Recently in several papers, Balog, Erdös, Maier, Sárközy, and Stewart have studied problems of the following type: if $A_1, \dots, A_k$ are "dense" sets of positive integers, then what can be said about the arithmetical properties of the sums $a_1 + \cdots + a_k$ with $a_1 \in A_1, \dots, a_k \in A_k$? In particular, Balog and Sárközy proved that there is a sum a_1+a_2 ($a_1 \in A_1$, $a_2 \in A_2$) for which $P(a_1+a_2)$ is "small", i.e., all the prime factors of a_1+a_2 are small. On the other hand, Balog and Sárközy and Sárközy and Stewart studied the existence of a sum $a_1 + \cdots + a_k$ for which $P(a_1 + \cdots + a_k)$ is large.

In this paper we study $p(a_1 + \cdots + a_k)$. Our goal is to show that if $A_1, \dots, A_k$ are sets of positive integers then there exists a sum $a_1 + \cdots + a_k$ with $a_1 \in A_1, \dots, a_k \in A_k$ that is divisible by a "small" prime. In the most interesting special case, namely $A_1 = \cdots = A_k$, there are sums $a_1 + \cdots + a_k$ divisible by k, so that $p(a_1 + \cdots + a_k) \le k$. In order to exclude such trivial cases, we shall ask that the "small" prime factor of $a_1 + \cdots + a_k$ also exceeds some prescribed bound V.

In §3 we will study the case when the geometric mean of the cardinalities of the sets $A_i \subset \{1, \dots, N\}$ is between $\sqrt{N}$ and N. The crucial tool will be the large sieve. In §4 we will extend the range (when $k > 2$) by studying the case $\min_i |A_i| > N^{1/k+\varepsilon}$. Here Gallagher's larger sieve will be used. The results in §§3 and 4 do not give especially good results when the sets $A_1, \dots, A_k$ are very "dense". In §5, we will give an essentially best possible result for the small prime factors of the sums $a_1 + \cdots + a_k$ in the case when

$$(|A_1| \cdots |A_k|)^{1/k} > N \exp(-c \log k \log N / \log\log N)$$

for a certain positive constant c. Finally in §§6 and 7 we will construct sets so that none of the sums $a_1 + \cdots + a_k$ has a small prime factor. In particular, in §6 we will discuss the conjecture of Ostmann [6] that