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INVARIANT SUBSPACES OF $\mathscr{H}^p$ FOR MULTIPLY CONNECTED REGIONS

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To David Lowdenslager, in memoriam

A closed linear subspace of $\mathscr{H}^p(G)$ is said to be *invariant* if $zf(z)$ is in $\mathscr{M}$ for all $f(z) \in \mathscr{M}$. It is said to be fully invariant if $r(z)f(z)$ is in $\mathscr{M}$ for all $f \in \mathscr{M}$ and all rational functions $r(z)$ with poles in the complement of G. This paper investigates those invariant subspaces of $\mathscr{H}^p(G)$, for a multiply connected G, which are invariant but not fully invariant. We show that an invariant subspace $\mathscr{M}$ fails to be fully invariant if and only if there is one bounded component G_i of the complement of $\overline{G}$ such that the ratio of any two functions in $\mathscr{M}$ has a pseudo-continuation to a meromorphic function in the Nevanlinna class of G_i. This allows us to give a complete characterization of those invariant subspaces of $\mathscr{H}^p(G)$ which contain the constants.

0. Introduction. Let G be a finitely connected bounded domain in C with smooth boundary contours. It is the purpose of this paper to study the closed linear subspaces of $\mathscr{H}^p(G)$ which are invariant under multiplication by z, that is, those subspaces $\mathscr{M}$ such that $zf(z)$ is in $\mathscr{M}$ whenever $f(z)$ is. The study of such spaces was initiated by Beurling [1] who gave a complete characterization of the invariant subspaces of $\mathscr{H}^2(\Delta)$, where Δ is the unit disk. Shortly thereafter Helson and Lowdenslager also investigated further problems of invariant subspaces using Beurling's methods.

Beurling's characterization is not difficult to extend to general simply connected domains, but the problem of characterizing the invariant subspaces for multiply connected domains is more complicated. A subspace $\mathscr{M}$ of $\mathscr{H}^p(G)$ is said to be fully invariant if it is invariant under multiplication by rational functions whose poles are in the complement of $\overline{G}$. For simply connected domains all invariant subspaces are fully invariant, but this is no longer true for multiply connected G.

It is possible to give a characterization similar to Beurling's for the *fully* invariant subspaces of $\mathscr{H}^p(G)$. This was carried out for the annulus by Sarason [12] and for more general domains by Hasumi [5] and Voichick [13], [14].