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ON THE GLOBAL DIMENSION OF FIBRE PRODUCTS

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In this paper we will sharpen Wiseman's upper bound on the global dimension of a fibre product [Theorem 2] and use our bound to compute the global dimension of some examples. Our upper bound is used to prove a new change of rings theorem [Corollary 4]. Lower bounds on the global dimension of a fibre product seem more difficult; we obtain a result [Proposition 12] which allows us to compute lower bounds in some special cases.

A commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R' \end{array}$$

is said to be a *Cartesian* square if given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there exists a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. We will assume that j_2 is a surjection so that results of Milnor [**M**] apply. The ring R is called a *fibre product* (or pullback) of R_1 and R_2 over R'.

The homological properties of a fibre product R have been studied previously. Milnor [**M**, Chapter 2] has characterized projective modules over such a ring R. Facchini and Vamos [**FV**] have obtained analogues of Milnor's theorems for injective and flat modules. Wiseman [**W**] has used Milnor's results to obtain an upper bound on $\operatorname{lgldim} R$; in particular, Wiseman's results show that R has finite left global dimension whenever the rings R_i have finite left global dimension and $\operatorname{fd}(R_i)_R$ are both finite, where $\operatorname{fd}(R_i)_R$ represents the flat dimension of R_i as a right R-module. Vasconcelos [**V**, Chapters 3 and 4] and Greenberg [**G1** and **G2**] have studied commutative rings of finite global dimension which are fibre products and have used their results to classify commutative rings of global dimension 2. Osofsky's example of a commutative local ring of finite global dimension having zero divisors can be described as a fibre product (see [**V**, p. 29–30]). Fibre products