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FUNCTIONS IN $R^2(E)$ AND POINTS OF THE FINE INTERIOR

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Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In this paper we study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. We also prove that if $R^2(E) \neq L^2(E)$, there is a subset $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

1. Introduction. Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In **[16]** we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at bounded point evaluations. The case $p = 2$ is different. Fernström has shown in [7] that $R^2(E)$ can be unequal to $L^2(E)$ without there being any bounded point evaluations for $R^2(E)$. In this paper we use the fine topology introduced by Cartan to study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. Finely open sets of this kind are contained in certain "Swiss cheese sets". We also prove that if $R^2(E) \neq L^2(E)$, there is a set $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.