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THEORY OF BOUNDED GROUPS AND THEIR BOUNDED COHOMOLOGY

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Bounded cohomology H_b^* can be defined for groups and for topological spaces. Recent work has shown that $H_b^*(M)$ of a topological space M depends only on $\Pi_1(M)$. In this paper we consider a new concept—a bounded group—and thereby expand the definition of bounded cohomology. We prove that bounded cohomology groups are themselves bounded groups and develop their properties in lower dimensions. In particular, elements of $H_b^2(G, A)$ classify bounded group extensions of G by A. As an application of the theory of bounded groups we construct the Lyndon spectral sequence. The result obtained is Theorem 3, which states that $H_b^n(H, A)^G \cong_b H_b^n(G, A)$, when G/H is finite.

1. Definitions and observations. We introduce the ideas of a bounded group and a bounded group homomorphism. We want to ensure that the mappings $(x, y) \to xy$ and $x \to x^{-1}$ are themselves bounded homomorphisms. Thus, we define a *norm* (actually a pseudo-norm) on a group G to be a function $\| \ \|: G \to R$ satisfying, for non-negative constants M, c, M', c',

(i) $\|x\| \geq 0$, for all x in G,
(ii) $\|xy\| \leq M(\|x\| + \|y\|) + c$,
(iii) $\|x^{-1}\| \leq M'\|x\| + c'$.

G together with its norm is a *bounded group*. A homomorphism f between two bounded groups G and H is *bounded* if and only if there exist non-negative constants M_f and c_f such that

$$\|f(g)\| \leq M_f\|g\| + c_f, \quad \text{for all } g \text{ in } G.$$

Two norms on G are *equivalent* if there exists a bounded isomorphism f between the two bounded groups; i.e., both f and f^{-1} must be bounded. The symbol for a bounded isomorphism will be $\cong_b$. It is worth noting that the composition of two bounded homomorphisms is bounded.

The cross-product of two bounded groups, $G \times H$, is a bounded group under the natural norm defined as follows:

$$\|(g, h)\| = \|g\| + \|h\|.$$