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DIFFERENTIAL IDENTITIES, LIE IDEALS, AND POSNER'S THEOREMS

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Two well-known results of E. C. Posner state that the composition of two nonzero derivations of a prime ring cannot be a nonzero derivation, and that in a prime ring, if the commutator of each element and its image under a nonzero derivation is central, then the ring is commutative. Our purpose is to show how the theory of differential identities can be used to obtain these results and their generalizations to Lie ideals and to rings with involution.

A number of authors have generalized these theorems of Posner in several ways. To be more specific, let R be a prime ring with center Z, and let d and h be derivations of R. The specific statements of Posner's theorems, to which we shall refer frequently, are the following:

POSNER'S FIRST THEOREM [**25**; Theorem 1, p. 1094]. *If* $\operatorname{char} R \neq 2$ *and if the composition* dh *is a derivation of* R, *then either* $d = 0$ *or* $h = 0$.

POSNER'S SECOND THEOREM [**25**; Theorem 2, p. 1097]. *If* $xx^d - x^d x \in Z$ *for all* $x \in R$, *then either* $d = 0$ *or* R *is commutative.*

The proof of the first theorem is fairly easy and extends to ideals of R. For this theorem, the case when $\operatorname{char} R = 2$ was obtained in [**6**] and later in [**13**], which also gives some generalizations to the case when $\operatorname{char} R \neq 2$ and R is a semi-prime ring. No attempt seems to have been made to extend Posner's first theorem to a Lie ideal L of R, assuming that dh is a Lie derivation on L. Several authors (see [**5**], [**7**], [**8**], [**16**], and [**22**]) have shown that $d = 0$ or $h = 0$ when $L^{dh} = 0$ or $L^{dh} \subset Z$. The second theorem of Posner was much more difficult to prove than the first, although an easier proof has been found [**3**]. When $\operatorname{char} R = 2$, this result is easy to prove. One such proof appears in [**1**] and, although not stated, it holds for Lie ideals of R. Partial generalizations of Posner's second theorem to ideals [**10**] and to Lie ideals when $\operatorname{char} R \neq 2$ [**4**] have also been obtained. More recently, a full generalization to Lie ideals when $\operatorname{char} R \neq 2$ has been proved