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SOME RESULTS ON SPECKER'S PROBLEM

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Say that the Specker Property holds for a well ordered cardinal $\aleph$, and write this as SP($\aleph$), if the power set of $\aleph$ can be written as a countable union of sets of cardinality $\aleph$. Specker's Problem asks whether it is possible to have a model in which SP($\aleph$) holds for every $\aleph$. In this paper, we construct two models in which the Specker Property holds for a large class of cardinals. In the first model, SP($\aleph$) holds for every successor $\aleph$. In the second model, SP($\aleph$) holds for every limit $\aleph$ and for certain successor $\aleph$'s.

In 1957, Specker **[13]** stated the following question (which will henceforth be referred to as Specker's Problem): Is it consistent with the axioms of ZF to have, for each ordinal α, a countable sequence $\langle A_n : n < \omega \rangle$ of subsets of $2^{\aleph_\alpha}$ so that $|A_n| = \aleph_\alpha$ for all n and $2^{\aleph_\alpha} = \bigcup_{n\in\omega} A_n$? Since the existence of one ordinal α so that $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$ implies that $\aleph_{\alpha+1}$ is singular, a model in which the above holds would be one in which the Axiom of Choice is false. Indeed, it can easily be seen that in such a model, AC_ω is false.

Lévy **[9]**, shortly after the invention by Cohen of forcing, constructed a model in which $2^{\aleph_0}$ is a countable union of countable sets. A later result on Specker's Problem was obtained in **[6]**, in which it was shown that, relative to the existence of a proper class of strongly compact cardinals, it is consistent for every infinite set to be a countable union of sets of smaller cardinality.

Unfortunately, we still do not know whether Specker's Problem is consistent. In this paper, we will prove the following two theorems, each of which provides a partial answer to Specker's Problem for a large class of cardinals.

THEOREM 1. Con(ZFC + *There exists a regular limit of supercompact cardinals*) $\Rightarrow$ Con(ZF + *For every successor ordinal* α, $2^{\aleph_\alpha}$ *is a countable union of sets of cardinality* $\aleph_\alpha$).

THEOREM 2. Con(ZFC + GCH + *There is a cardinal* κ *which is* $2^{2^{[\kappa^{+\omega}]^{<\omega}}}$ *supercompact*) $\Rightarrow$ Con(ZF + *For every limit ordinal* λ, $2^{\aleph_\lambda}$ *is a*