PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 1, 1988

SZEGÖ'S CONJECTURE ON LEBESGUE CONSTANTS FOR LEGENDRE SERIES

C. K. QU AND R. WONG

In 1926, Szegö conjectured that the Lebesgue constants for Legendre series form a monotonically increasing sequence. In this paper, we prove that his conjecture is true. Our method is based on an asymptotic expansion together with an explicit error bound, and makes use of some recent results of Baratella and Gatteschi concerning uniform asymptotic approximations of the Jacobi polynomials.

1. Introduction. The Lebesgue constants for classical Fourier series are defined by

$$(1.1)\qquad \rho_n = \frac{1}{\pi}\int_0^\pi \frac{|\sin(n+1/2)t|}{\sin(t/2)}\,dt, \qquad n = 1, 2, 3, \ldots;$$

see [**18**, p. 172]. Fejer [**4**] was the first to show that

$$(1.2)\qquad \rho_n = \frac{4}{\pi^2}\log n + c_0 + \frac{c_1}{n} + \frac{\alpha(n)}{n^2},$$

where c_0 and c_1 are constants and $\alpha(n)$ is bounded for all n. From (1.2), he deduced that

$$(1.3)\qquad \rho_{n+1} - \rho_n > 0$$

for large n. He further conjectured that (1.3) holds for all $n \geq 1$, a conjecture later proved by Gronwall [**7**]. Gronwall's result was considerably improved by Szegö [**12**], who showed that the sequence of differences of the Lebesgue constants ρ_n is in fact completely monotonic, i.e., $\Delta\rho_n = \rho_{n+1} - \rho_n > 0$ and $(-1)^{r-1}\Delta^r\rho_n > 0$ for $r = 2, 3, \ldots$.

In exactly the same manner, one can investigate the properties of the Lebesgue constants

$$(1.4)\qquad L_n = \frac{n+1}{2}\int_{-1}^{1} |P_n^{(1,0)}(x)|\,dx = (n+1)\int_0^\pi \sin\frac{\theta}{2}\cos\frac{\theta}{2}|P_n^{(1,0)}(\cos\theta)|\,d\theta, \qquad n = 1, 2, \ldots,$$

for Legendre series at $x = 1$, where $P_n^{(1,0)}(x)$ is the Jacobi polynomial with $\alpha = 1$ and $\beta = 0$. The asymptotic formula

$$(1.5)\qquad L_n = \frac{2^{3/2}}{\sqrt{\pi}}n^{1/2} + o(n^{1/2}), \qquad n \to \infty,$$