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THE GEOMETRY OF SUM-PRESERVING PERMUTATIONS

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Geometric characterizations of the semigroup of permutations which preserve convergence of series are presented.

1. Introduction. A well-known result of Riemann asserts that the sum of a conditionally convergent series may be changed to any value by suitably permuting the terms of the series. We introduce geometric tools to prove, among other results, that the set S of permutations which do not change the value of any convergent series is exactly the set of permutations which fix a type of asymptotic density of subsets of the natural numbers (Theorem 1.6).

Several authors ([**Le**], [**A**], see Schafer's survey article [**Sch**]) have given characterizations of the set S of permutations. The ideas of Levi [**Le**] are combinatoric in nature. Those of Agnew, on the other hand, come from the theory of summability of series; see e.g. Chapter III of [**H**], especially Theorems 1–3 of Schur and Toeplitz. In fact, consideration of Theorem 1 leads to certain geometric notions (Definitions 1.2 and 2.1) with which we express our characterizations of S.

1.1. NOTATION. If (a_i) is a sequence of real numbers, $\sum a_i$ denotes the limit of partial sums $\lim_{n\to\infty}\sum_{k=1}^{n} a_i$. Let $\mathscr{C} = \{(a_i) \colon \sum a_i \in \mathbf{R}\}$ be the set of convergent series.

Let P denote the group of permutations of the natural numbers $N = \{1, 2, \dots\}$. If $\sigma \in P$ and (a_i) is a sequence, $\sigma(a_i)$ is the sequence whose ith term is $a_{\sigma^{-1}(i)}$. Let

$$S^* = \{\sigma \in P \colon (a_i) \in \mathscr{C} \text{ implies } \alpha(a_i) \in \mathscr{C}\},$$
$$S = \left\{\sigma \in S^* \colon (a_i) \in \mathscr{C} \text{ implies } \sum(a_i) = \sum \sigma(a_i)\right\}.$$

It is proven in [**Sch**] that $S = S^*$; we see this as Corollary 3.5. Note, however, that S and S^* are clearly semigroups.

Last, $\#X$ denotes the cardinality of a set X, and if $n, m \in \mathbf{N}$, $n \le m$, then $I(n, m) = \{n, n+1, \dots, m\}$.