PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 2, 1988

RELATIVE WIDTH MEASURES AND THE PLANK PROBLEM

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A relative width measure in a convex body K in R^n for a set 6 of directions is a Borel probability measure in K such that the measure of the intersection of K with each slab orthogonal to a direction in 6 is equal to the relative width of the slab. Such measures are studied in connection with the unsolved plank problem of Th. Bang.

0. Introduction. Tarski's plank problem was solved by Th. Bang [2] when he showed that if a convex body K in R^n is covered by a finite number of slabs, the sum of their widths is at least the minimum width of K. Bang conjectured that a stronger, and affine invariant, inequality should hold; namely, that the sum of the relative widths of the slabs is at least one (the relative width of a slab is its width divided by the width of K in the same direction). This is still unsolved.

A relative width measure is a Borel probability measure in K such that the measure of the intersection of K with any slab is precisely the relative width of the slab. An example, known to Achimedes, is normalized surface area measure in a ball in R^3; another is the projection of this measure, normalized, in a disc in R^2. If such a measure exists in K, then Bang's conjecture is true for K. This observation has been made several times in the literature, but does not seem to have been thoroughly investigated.

We study these measures, always with Bang's conjecture in mind. For this application, the measures need only have the relative width property for directions corresponding to the covering slabs, and in fact a reduction shows that we need only seek them for coordinate directions. Theorem 1 shows that measures with the latter property always exist in R^2, which generalizes the known special case of Bang's conjecture for two slabs. However, Example 2 shows that even measures with this weaker property do not generally exist for K in R^3.

Section 3 concerns measures with the relative width property for infinite sets of directions. Here, using Fourier transform techniques and particularly a method due to K. Falconer, we show (Theorem 3) that measures with the relative width property for all directions do