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ON THE ILIEFF-SENDOV CONJECTURE

JOHNNY E. BROWN

The well-known Ilieff-Sendov conjecture asserts that for any polynomial $p(z) = \prod_{k=1}^{n}(z-z_k)$ with $|z_k| \le 1$, each of the disks $|z-z_k| \le 1$ $(1 \le k \le n)$ must contain a critical point of p. This conjecture is proved for polynomials of arbitrary degree n with at most four distinct zeros. This extends a result of Saff and Twomey.

1. Introduction. The Gauss-Lucas Theorem states that all the critical points of a polynomial $p(z)$ lie in the convex hull of its zeros. This is a result concerning the position of all the zeros of $p'(z)$ relative to all the zeros of $p(z)$. Suppose we focus attention on any arbitrarily fixed zero of $p(z)$ and ask for the location of a zero of $p'(z)$ relative to it. This leads to the well-known conjecture of Ilieff and Sendov [3; Problem 4.5] which asserts that if $p(z)$ has the form

$$p(z) = \prod_{k=1}^{n}(z-z_k), \qquad |z_k| \le 1 \quad (1 \le k \le n) \tag{1}$$

then each of the disks $|z - z_k| \le 1$ $(1 \le k \le n)$ contains a zero of $p'(z)$. The polynomial $p(z) = z^n - 1$ shows that "1" is sharp. This conjecture is nearly a quarter of a century old and has been verified in some special cases, most notably if $p(z)$ has the form (1) and if

(A) $2 \le n \le 5$ [1, 6, 8],

(B) $p(0) = 0$ [10],

(C) $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $a_k \le 0$ $(0 \le k \le n-1)$ [11],

(D) $p(z)$ has only real zeros [7],

(E) $p(z) = z^{n_3} + a_2 z^{n_2} + a_1 z^{n_1} + a_0 z^{n_0}$ $(n_0 < n_1 < n_2 < n_3)$ or $p(z) = \sum_{j=0}^{m} a_j z^{n_j}$ $(n_0 \le n_1 < \cdots < n_m)$ [12],

(F) the vertices of the convex hull of the zeros of $p(z)$ all lie on $|z| = 1$ [10],

(G) the convex hull of the zeros of $p(z)$ is a triangular region [12],

(H) $p(z) = (z - z_1)^{n_1}(z - z_2)^{n_2}(z - z_3)^{n_3}$ [9].

The last case (H), due to Saff and Twomey, states that the conjecture is true for any polynomial of the form (1) with at most three distinct