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q-BETA INTEGRALS AND THE q-HERMITE POLYNOMIALS

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The continuous q-Hermite polynomials are used to give a new proof of a q-beta integral which is an extension of the Askey-Wilson integral. Multilinear generating functions, some due to Carlitz, are also established.

1. Introduction. Let $q \in (-1, 1)$ and define the q-shifted factorials by

$$(a)_0 = (a;q)_0 = 1,$$
$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n = 1, 2, \dots,$$
$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Basic hypergeometric series are defined by

$$ {}_{r+1}\phi_r(a_1, a_2, \dots, a_{r+1}; b_1, b_2, \dots, b_r; z) \equiv {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} \middle| \, z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_{r+1})_n}{(q)_n(b_1)_n\cdots(b_r)_n} z^n.$$

The continuous q-Hermite polynomials $\{H_n(x|q)\}$ are given by

$$H_n(\cos\theta|q) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k(q)_{n-k}} e^{i(n-2k)\theta} \tag{1.1}$$

(see [2]). Their orthogonality [2, 3] is

$$\int_0^\pi w(\theta) H_m(\cos\theta|q) H_n(\cos\theta|q)\, d\theta = (q;q)_n \delta_{nm} \tag{1.2}$$

where

$$w(\theta) = \frac{(q)_\infty}{2\pi}(e^{2i\theta})_\infty(e^{-2i\theta})_\infty. \tag{1.3}$$

Rogers also introduced the continuous q-ultraspherical polynomials $\{C_n(x;\beta|q)\}$ generated by

$$\sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q) t^n = \frac{(\beta t e^{i\theta})_\infty(\beta t e^{-i\theta})_\infty}{(te^{i\theta})_\infty(te^{-i\theta})_\infty} \tag{1.4}$$