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HARMONIC GAUSS MAPS

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A construction is given whereby a Riemannian manifold induces a Riemannian metric on the total space of a large class of fibre bundles over it. Using this metric on the appropriate bundles, necessary and sufficient conditions are given for the Gauss map and the spherical Gauss map to be harmonic. A weak maximum principle is applied to the Gauss map of an isometric immersion into Euclidean space in order to prove a sufficient condition for when such an immersion with parallel mean curvature vector must be minimal.

1. Introduction. For a Riemannian manifold M^m and an isometric immersion $f\colon M \to \mathbf{R}^n$, Ruh-Vilms **[13]** proved that the Gauss map of f is harmonic if and only if f has parallel mean curvature vector. Here the Gauss map assigns to a point $p \in M$ the m-dimensional subspace of $\mathbf{R}^n$ obtained from the parallel translation of f_*T_pM to the origin. It thus takes values in the Grassmannian $G_m(n)$, endowed with an $O(n)$-invariant Riemannian metric.

In this paper we generalize the Ruh-Vilms theorem to isometric immersions $f\colon M \to N$, where N^n is a Riemannian manifold. There are two natural ways in which a Gauss map can be defined. The first, which we call simply the Gauss map, $\gamma_f\colon M \to G_m(TN)$, sends a point $p \in M$ to the tangent m-plane f_*T_pM in the Grassmann bundle of tangent m-planes of N. The second, which we call the spherical Gauss map, $\nu_f\colon TM_1^\perp \to TN_1$, maps a unit normal vector of M to itself as a unit tangent vector of N.

The notion of harmonicity of these Gauss maps requires some Riemannian metric on the, generally non-trivial, fibre bundles $G_m(TN)$, $TM_1^\perp$, and TN_1. In §2 we present a natural construction of a Riemannian metric on the total space of a large class of fibre bundles over a Riemannian manifold. As an immediate application of this construction we analyse the geometry of this metric on the tangent bundle, where the Sasaki metric is obtained. To illustrate our formalism we give a global version of Raychauduri's equation on the tangent bundle level.

Using the metrics constructed by this method, we are then able to prove a generalized Ruh-Vilms theorem for the Gauss map γ_f in