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RELATIVE DIMENSION, TOWERS OF PROJECTIONS AND COMMUTING SQUARES OF SUBFACTORS

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Dedicated to the memory of Henry Dye

We study the set of projections of the type II_1 factor M which expected on the subfactor $N \subset M$ are scalar multiples of the identity. The set of all these scalars, denoted $\Lambda(M, N)$, is an invariant for the inclusion $N \subset M$. We compute $\Lambda(M, N)$ when $[M : N] < 4$, when N is locally trivial and some parts of $\Lambda(M, N)$ when $[M : N] > 4$. We prove that projections expected on the same scalar in N are conjugate by a unitary element in N. We apply all these to the commuting square problem.

Introduction. In a type II_1 factor a projection may have any dimension between 0 and 1. This corresponds to the fact that Hilbert modules H over a type II_1 factor M may have any positive number as (relative) dimension $\dim_M H$ **[9]**.

There has been more and more evidence in the past ten years or so that it is much more useful to regard a type II_1 factor M together with its subalgebras N and more generally to consider pairs of arbitrary algebras M, N. The corresponding appropriate notion of module is then the one introduced by Connes in **[2]**, the $N - M$ Hilbert bimodules (or correspondences).

If $N \subset M$ is a subfactor of the type II_1 factor M then V. Jones had the idea to consider the number

$$\dim_N H / \dim_M H \ (= \dim_N H \dim_{M'} H)$$

as an invariant up the conjugacy by automorphisms of M for the subfactor N (this number is independent of H by **[9]**). Jones called this number the index of N in M denoting it $[M : N]$. One of his remarkable results in **[6]** is that $[M : N]$ can only take the values $\{4\cos^2 \pi/(n+2) | n \geq 0\} \cup [4, \infty]$.

The number $[M : N]$ can also be interpreted in a more intrinsic way: it is the dimension of the smallest nonzero projection in M which expected on N is a scalar multiple of the identity (by **[6]** and **[12]**). This is somehow related to the fact that $[M : N]$ can be viewed as the minimal possible dimension $\dim_{M,N} H \overset{\text{def}}{=} \dim_M H \dim_N H$ (**[14]**)