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REMARKS ON THE DIFFERENTIAL ENVELOPES OF ASSOCIATIVE ALGEBRAS

R. COQUEREAUX AND D. KASTLER

Dedicated to the memory of Henry Dye

We study the relationship between the two types of differential envelopes of Z/2-graded associative complex algebras; and describe the differential envelopes of a Z/2-graded algebra as a simple modification of those of the associated ungraded algebra.

In this note we describe some aspects of the two existing notions of differential envelope of a $\mathbf{Z}/2$-graded complex (or, for that matter real) associative algebra. Both notions are basic in the non commutative geometry under current development.

In §1, which fixes notation, we recall the definitions and main properties of the two types of differential envelope (= universal differential algebra) one attaches to a $\mathbf{Z}/2$-graded algebra A. The first, $\tilde{\Omega}(A) = \mathbf{C}\tilde{\mathbf{1}} \oplus \Omega(A)$, pertains to general algebras (unital or not), and has a differential which vanishes nowhere on A (in particular, if A is unital with unit $\mathbf{1}$, $d\mathbf{1} \neq 0$). The second, ΩA, is defined only for unital algebras, and has a differential vanishing on the unit. $\tilde{\Omega}(A)$ can accordingly be directly defined in terms of tensors over A, whilst the construction of ΩA uses either tensors over an A-bimodule, or a subfamily of tensors over A. Both $\tilde{\Omega}(A)$ (for that matter $\Omega(A)$) and ΩA are universal objects (= tied up with functors), the first in the general and the second in the unital category.

In §2 we discuss the relationship between the two notions, each of which is definable in terms of the other. On the one hand one has $\tilde{\Omega}(A) = \Omega\tilde{A}, \tilde{A} = \mathbf{C}\tilde{\mathbf{1}} \oplus A$ the augmented ($\mathbf{Z}/2$-graded) algebra. On the other hand there is an injective multiplicative linear map of ΩA into the right ideal $A\tilde{\Omega}(A)$ of $\tilde{\Omega}(A)$, allowing one to consider ΩA as a linear subspace of tensors over A (vanishing under the "consecutive diagonal mappings", and multiplying under "concatenation").

In §3 we show how the differential envelopes of a $\mathbf{Z}/2$-graded algebra A can be constructed in terms of those of the associated ungraded algebra (merely through a simple modification of the latter's product).