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THE LATTICE OF PSEUDOVARIETIES OF INVERSE SEMIGROUPS

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As introduced by Eilenberg and Schützenberger, a pseudovariety is a class of finite algebras closed under the formation of homomorphic images, subalgebras and direct products of finitely many algebras. Many previous results about the lattice of varieties of inverse semigroups are found to have analogues for the lattice of pseudovarieties of (finite) inverse semigroups. In particular, certain intervals are modular, including the interval consisting of the pseudovarieties of groups.

1. Introduction and summary. A *pseudovariety*, introduced by Eilenberg and Schützenberger [**8**], is a class of finite algebras closed under forming homomorphic images, subalgebras, and finite direct products (each class of algebras considered will be assumed to consist of algebras all of the same type). As usual, we consider inverse semigroups to have two operations, that of multiplication, and that of inversion, so that just as the class $\mathscr{I}$ of all inverse semigroups is a variety, the class of all finite inverse semigroups $\mathscr{I}_F$ is a pseudovariety.

Finite semigroups have long been a part of language and automata theory (Eilenberg [7], Lallement [13]), since each recognizable language has a finite syntactic semigroup and each automaton has a finite action semigroup. Not only do pseudovarieties give a natural way of classifying finite algebras generally, but also pseudovarieties of semigroups are in a natural one-to-one correspondence with varieties of recognizable (or rational) languages [7, Vol. B, Theorem 3.4s].

Finite inverse semigroups and injective (or reversible) automata are now also studied in automata theory, for example in [**10**, **12**, **14**, **15**, **16**, **19**, **21**]. The pseudovariety of semigroups generated by $\mathscr{I}_F$ was shown by Ash [**3**] to be the pseudovariety of (finite) semigroups with all idempotents commuting, which we denote by $\mathscr{S}_{ic,F}$. The variety of languages corresponding to $\mathscr{S}_{ic,F}$ has been described by Margolis and Pin ([**15**, Theorem 5.2] and [**16**]). Of course each subpseudovariety of $\mathscr{S}_{ic,F}$ also corresponds to a variety of languages, and this correspondence has in part been studied by Ash, Hall and Pin [**4**]. Consider a map $\alpha: \mathscr{L}_{spv}(\mathscr{S}_{ic,F}) \to \mathscr{L}_{pv}(\mathscr{I}_F)$, given by $\mathscr{P}\alpha = \mathscr{P} \cap \mathscr{I}_F$ (here $\mathscr{L}_{spv}(\mathscr{S}_{ic,F})$ denotes the lattice of (semigroup) pseudovarieties