PACIFIC JOURNAL OF MATHEMATICS
Vol. 138, No. 1, 1989

STATE EXTENSIONS AND A RADON-NIKODYM THEOREM FOR CONDITIONAL EXPECTATIONS ON VON NEUMANN ALGEBRAS

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Let M be a von Neumann algebra with a von Neumann subalgebra M_0. If E is a conditional expectation (i.e., projection of norm one) from M into M_0, then any faithful normal state φ_0 admits a natural extension $\varphi_0 \circ E$ with respect to E in the sense that $E = E_{\varphi_0 \cdot E}$. If E_ω is only an ω-conditional expectation, then $\varphi_0 \circ E_\omega$ is not always an extension of φ_0. This paper is devoted to the construction of an extension $\tilde{\varphi}_0$ of φ_0 generalizing the above situation for ω-conditional expectations, which leads also to a Radon-Nikodym theorem for ω-conditional expectation under suitable majorization condition.

Let M be a von Neumann algebra with a faithful normal state ω and M_0 a von Neumann subalgebra of M. A conditional expectation of M onto M_0 leaving ω invariant exists if and only if M_0 is stable under the modular group σ^ω. This is a result of Takesaki ([**15**], 10.1) and it was the reason for a generalized conditional expectation $E_\omega \colon M \to M_0$, which always exists and is referred to as the ω-conditional expectation, to be introduced by Accardi and Cecchini ([**1**]). If E_ω is actually a projection, then for a faithful normal state φ_0 on M_0 the composition $\tilde{\varphi}_0 = \varphi_0 \circ E_\omega$ is a natural extension of φ_0 to M and $E_\omega = E_{\tilde{\varphi}}$. In general, $\varphi_0 \circ E_\omega$ is not an extension of φ_0 and as a consequence of Theorem 4 in [**11**] (see also [**12**]) there is no extension of φ_0 possessing the same generalized conditional expectation mapping as ω. We give a construction of a $\tilde{\varphi}_0$ that can be described briefly as follows.

Assuming that $M \subset B(H)$ and ω is determined by a cyclic and separating vector $\Omega \in H$, we consider the restriction of the action of M_0 to $[M_0\Omega] = H_0$. There is a natural positive cone $P_0 \subset H_0$ with respect to M_0 such that $\omega|M_0$ and φ_0 have the vector representatives Ω and Φ_0 in P_0, respectively. We say that the vector state $\tilde{\varphi}_0(a) = \langle a\Phi_0, \Phi_0\rangle$ is the canonical extension of φ_0 with respect to ω. If the cocycle $[D\varphi_0, D(\omega|M_0)]_t$ is in the fixed point algebra of E_ω, then our $\tilde{\varphi}_0$ reduces to $\varphi_0 \circ E_\omega$, and of course, this is the case where E_ω is a projection. In fact, $\tilde{\varphi}_0$ depends rather on E_ω than ω itself; that is, if